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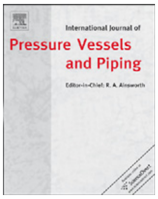

A three-dimensional layerwise-differential quadrature free vibration analysis of laminated cylindrical shells

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ABSTRACT

A mixed layerwise theory and differential quadrature (DQ) method (LW-DQ) for three-dimensional free vibration analysis of arbitrary laminated circular cylindrical shells is introduced. Using the layerwise theory in conjunction with the three-dimensional form of Hamilton's principle, the transversely discretized equations of motion and the related boundary conditions are obtained. Then, the DQ method is employed to discretize the resulting equations in the axial directions. The fast convergence behavior of the method is demonstrated and its accuracy is verified by comparing the results with those of other shell theories obtained using conventional methods and also with those of ANSYS software. In the case of arbitrary laminated shells with simply supported ends, the exact solution is developed for comparison purposes. It is shown that using few DQ grid points, converged accurate solutions are obtained. Less computational efforts of the proposed approach with respect to ANSYS software is shown.

1. Introduction

Laminated composite materials are widely used in many branches of engineering, because of their strength to weight ratio, thus accurate assessment of their behavior under static and dynamic loading are needed. Various theories are used for analyzing composite laminated circular cylinders. Many of shell studies are based on the Kirchhoff–Love's hypothesis [1–6]. These theories produce accurate results for shells with large length-to-thickness and radius-to-thickness ratios, the so-called thin shell, or for shells with low material anisotropy. The application of such theories to moderately thick or thick composite shells can lead to serious errors. Using the first- and higher-order shear deformation theories, some modifications are done on the shell theories to include the effects of transverse shear deformation [7–12]. In order to accurately predict the natural frequencies of thick laminated shells, one should use the three-dimensional elasticity theory to account the transverse shear deformation effects. However, due to complexity of the governing equations, a powerful numerical technique is necessary for solving the resulting equations for arbitrary laminated thick shells with general boundary conditions.

A comprehensive survey of the early works dealing with three-dimensional vibration analysis of cylinders can be found in the review paper of Soldatos [13]. The research works on the vibration analysis of isotropic and composite cylindrical shells based on three-dimensional elasticity theory are increasingly continued [14–19]. In the previous researches, the finite element method, the Ritz method and the series solutions are the most popular methods used for vibration analysis of cylindrical shell. Due to the numerous applications of these shells in industry, however, search continues for additional powerful methods.

Because of intrinsic complexity of the problem based on the three-dimensional elasticity, exact solutions are not available for arbitrary laminated cylindrical shells with general boundary conditions. Hence, in this study, based on the three-dimensional theory of elasticity, a mixed layerwise-differential quadrature (LW-DQ) method for free vibration analysis of laminated composite circular cylindrical shell with arbitrary lamination scheme and boundary conditions is developed. It should be mentioned that other new numerical methods such as discrete singular convolution method can be used for free vibration analysis of shells [20,21].

The layerwise theory is a refined theory that can into account the thickness effects with minimum computational cost [22]. Unlike the equivalent single-layer theories, the layerwise theories assume separate displacement field expansions within each

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subdivision. Hence, the layerwise theory provides a kinematically correct representation of the strain field in discrete layers [22]. It shows an advantage of the layerwise methods in comparison with higher-order shear deformation theories. Since the shear strains are discontinuous, this leaves the possibility of the continuous transverse stresses between adjacent layers in the layerwise theory.

Differential quadrature method (DQM) as a numerical technique was used in structural analysis since 1988. A review of the early developments in DQM can be found in the papers by Bert and Malik [23,24]. The method has been widely used for static and free vibration analysis of beams and plates [23–38]. Liew and his co-workers [36–38] developed a layerwised differential quadrature based approach for the three-dimensional analysis of composite plates. In application of DQM for different structural problems, it was concluded that highly accurate results with less computations can be obtained.

2. The basic formulations

The geometric configuration of a laminated composite circular cylindrical shell is shown in Fig. 1. The coordinate system is located at the end plane of the cylindrical shell where the z-axis directed along longitudinal axis. Mean radius of the shell is denoted by R, uniform thickness by h and cylinder length by L.

The elastic strain energy V and the kinetic energy T of the cylinder can be written, respectively as

$$V = \iiint_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}v$$

$$T = \frac{1}{2}\int_0^h \int_0^{2\pi} \int_{r_\mathrm{i}}^{r_\mathrm{o}} \rho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \qquad (1)$$

where ρ is the mass per unit volume. σ_{ij} and ε_{ij} are the components of stress and strain tensors, respectively. As obvious from the first part of Eq. (1), for brevity purpose, the indicial summation rule is used hereafter.

The constitutive relations for an arbitrary lamina of the shell can be written as follows:

$$\begin{Bmatrix} \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{rz} \\ \sigma_{\theta z} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{rr} \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{\theta z} \end{Bmatrix} \qquad (2)$$

where $[C] = [T][\bar{C}][T]^\mathrm{T}$, in which $[\bar{C}]$ is the principal material stiffness matrix and $[T]$ is the transformation matrix [39].

Based on the three-dimensional small deformation theory of elasticity, the strain–displacement relations can be expressed as

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r} + \frac{\partial v}{r\partial\theta}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z},$$
$$\varepsilon_{r\theta} = \frac{\partial u}{r\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad \varepsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{\partial w}{r\partial\theta} \qquad (3)$$

For closed cylindrical shells, the displacement components in the θ-direction can be expanded as

$$u(r,\theta,z,t) = \sum_{m=1}^{\infty} U_m(r,z,t)\cos(m\theta)$$
$$v(r,\theta,z,t) = \sum_{m=1}^{\infty} V_m(r,z,t)\sin(m\theta)$$
$$w(r,\theta,z,t) = \sum_{m=1}^{\infty} W_m(r,z,t)\cos(m\theta) \qquad (4)$$

where 'm' represents the circumferential wave number. In order to build a high degree of transverse discretization generality into the model, the layerwise laminate theory of Reddy [22] is used to introduce the following expansions for the displacement components in the radial direction:

$$U_m(r,z,t) = \sum_{i=1}^{N_r} U_{im}(z,t)\psi_i(r) = U_{im}(z,t)\psi_i(r)$$
$$V_m(r,z,t) = \sum_{i=1}^{N_r} V_{im}(z,t)\psi_i(r) = V_{im}(z,t)\psi_i(r)$$
$$W_m(r,z,t) = \sum_{i=1}^{N_r} W_{im}(z,t)\psi_i(r) = W_{im}(z,t)\psi_i(r)$$
$$i = 1,2,\ldots,N_r \qquad (5)$$

where $\psi_i(r)$ denote the global interpolation functions in the r-direction; U_{im}, V_{im} and W_{im} represent the displacement components of all points located on the ith mathematical layer defined by $r = r_i$ in the r-, θ-, and z-directions, respectively. Also, $N_\mathrm{r}[= (N_\mathrm{npl}-1)N_m+1]$ represents the total number of nodes through the thickness of the shell, which depends on the number of mathematical layers N_m and node per layer (N_npl) in the thickness direction.

In the present study, in each mathematical layer, one-dimensional Lagrange interpolation functions are used and hence the global interpolation function $\psi_i(r)$ can easily be obtained. The layerwise concept is very general in that the number of subdivisions can be greater than, equal to, or less than the number of material layers through the thickness. Any desired degree of displacement variation through the thickness is easily

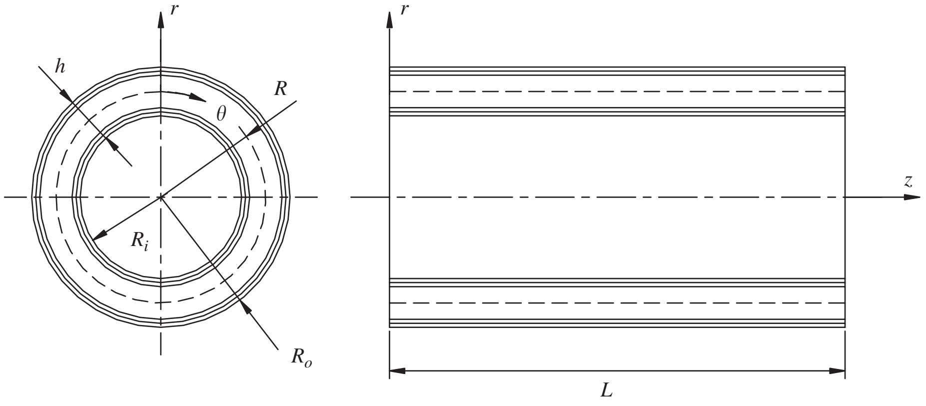

Fig. 1. Geometry of laminated circular cylindrical shell.