International Journal of Non-Linear Mechanics 61 (2014) 32–38

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

# The non-linear thickness-shear vibrations of quartz crystal plates under an electric field

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ARTICLE INFO

*Article history:*
Received 11 July 2012
Received in revised form
22 October 2013
Accepted 22 January 2014
Available online 1 February 2014

*Keywords:*
Thickness-shear vibration
Non-linear
Successive approximation
Plate
Resonator
Voltage

ABSTRACT

With the consideration of material and kinematic non-linearities, a non-linear system of two-dimensional equations for the strongly coupled thickness-shear and flexural vibrations of electroelastic plates is established by expanding mechanical displacements and electric potential into power series in the plate thickness coordinate and integrating over the thickness. Since the non-linear equations are too complicated to be solved directly by known methods, we utilized the Galerkin approximation to convert the non-linear equation of thickness-shear vibrations into an ordinary differential equation depending only on time by assuming the mode shape of linear vibrations. This non-linear forced vibration equation has been solved by the successive approximation method and we plotted frequency–response curves with different amplitude ratios and electrical voltages. Numerical results showed that the electric field has a more significant effect on vibration frequency compared with other known factors.



## 1. Introduction

Piezoelectric crystal resonators are key components of many frequency control and detection applications in various electronic products in our daily life. By the means of piezoelectric effect, the properly arranged electrode excites the plate into mechanical resonant vibrations that can provide a desired frequency which is needed in electronic circuits [1–3]. The linear two-dimensional plate theories for high frequency vibrations of piezoelectric plates have been extensively studied by Mindlin, Tiersten, Lee, and others with various approximate solutions including the resonance frequency, mechanical effects of electrodes, thermal effects, and even electrical parameters of crystal resonators [4–9]. With the growing demands for higher and more accurate frequency and the miniaturization of piezoelectric structures, noted non-linear phenomena such as derive level dependency (DLD) and activity dip have emerged along with the frequency instability of resonators [9,10]. Based on previous experimental and theoretical studies, it has been revealed that many causes such as material and kinematic non-linearities, electric fields, initial stresses, and size effects will cause frequency instability and affect the operation precision of piezoelectric resonators and structures [11–13]. Furthermore, earlier studies also showed that the effects of electric field in dielectric and piezoelectric solids are particularly complicated which even have effect on heat waves generation [14,15]. Consequently, the most important problem which challenges engineers and researchers is how to find the dominant factors of frequency variation of structures under applied loading conditions.

With this objective in mind, many researchers have employed analytical or numerical methods to study different non-linear effects on the frequency shift of thickness-shear vibrations of piezoelectric plates for various purposes. Yang and Guo [16] have investigated the effects of higher-order elastic constants on electromechanical coupling factors. Furthermore, Yang et al. [17,18] established two-dimensional equations for electroelastic plates and shells with relatively large shear deformation and obtained non-linear current amplitude–frequency curves near the resonance. Wu et al. [19] have studied electrically forced thickness-shear vibrations of quartz crystal plates with non-linear coupling to the extension and other modes with the finding that mode couplings affect energy trapping which should be restrained by selecting appropriate aspect ratios such as length to thickness. Besides analytical methods, numerical methods have also been tried. Wang et al. [20] have employed the finite element method to investigate the non-linear thickness-shear vibrations of quartz crystal plates. Based on the equations of non-linear three-dimensional piezoelectricity, Patel et al. [10] have studied well-known drive level dependence (DLD) phenomenon in quartz crystal resonators. The finite difference method has also been used for the vibration analysis with non-linear Mindlin plate equations [21].

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http://dx.doi.org/10.1016/j.ijnonlinmec.2014.01.010
0020-7462 

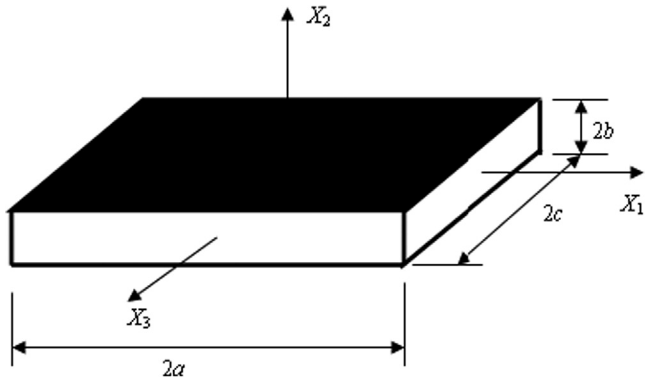


**Fig. 1.** A fully electroded AT-cut quartz crystal plate.

In our earlier studies, the non-linear Mindlin plate equations for strongly coupled thickness-shear and flexural vibrations with the consideration of material and kinematic non-linearities have been established [22]. Then the strongly coupled non-linear equation of thickness-shear vibrations has been solved by the combination of the Galerkin approximation and homotopy analysis method (HAM) which is a newly emerged technique for strong non-linear problems [23–25]. The amplitude–frequency relation we obtained indicated neither kinematic nor material non-linearities are the main factor of frequency shifts [23–25]. For this reason, we focus on the effect of electric field in a piezoelectric crystal plate. The general non-linear electroelastic equations were first presented by Tiersten and collaborators and have been used to analyze various problems of elastic plates with fully or partially electroded piezoelectric actuators and buckling of piezoelectric plates [26–29]. Tiersten's equations have been lately expanded to include cubic electric non-linearity by Yang [28].

In this paper, we established two-dimensional equations for the thickness-shear and flexural vibrations of electroelastic plates under strong electric fields. With the consideration of cubic electric non-linearity, these equations are too complicated to be solved by known methods. For this reason, we only retained the first-order potential which relates to the voltage across the electrodes as shown in Fig. 1. Then the non-linear equation of thickness-shear vibrations has been solved again by the combination of the Galerkin approximation and successive approximation method. By plotting frequency–response curves for different amplitude ratios and voltages, we found that the electric field has a more important effect on frequency shift which can be directly used for the examination of non-linear behavior of resonators, as we intended to from the beginning of this study.

## 2. The non-linear electroelastic plate equations

The linear Mindlin plate equations have been widely used to precisely predict the thickness-shear vibration frequency, study mode couplings and mode conversion, and investigate thermal effect and other problems related to the resonator design and applications [5,6]. These results from these equations are generally accurate and acceptable in practical applications [7,8]. Then it is natural to derive the non-linear electroelastic plate equations based on the established procedure. As a demonstration, we assume mechanical displacements and electric potential as

$$u_j(x_1,x_2,x_3,t)=\sum_{n=0}^{\infty} u_j^{(n)}(x_1,x_3,t)x_2^n, \quad j=1,2,3, \quad n=0,1,2,3\ldots,$$

$$\phi(x_1,x_2,x_3,t)=\phi^{(0)}(x_1,x_3,t)+\frac{x_2}{2b}\phi^{(1)}(x_1,x_3,t) +\left(\frac{x_2^2}{b^2}-1\right)\phi^{(2)}(x_1,x_3,t)+\frac{x_2}{b}\left(\frac{x_2^2}{b^2}-1\right)\phi^{(3)}(x_1,x_3,t), \tag{1}$$

where $u_j^{(n)}$ and $\phi^{(m)}$ $(m=0, 1, 2, 3)$ are the $n$th-order displacements and the $m$th-order electric potentials, respectively. It should be mentioned that the electric potential expansion in (1) is basically taken from Yang and Tiersten [26,28], which is also similar with Wang's expansion with slightly different notations [30,31]. Among these electric potentials, $\phi^{(1)}$ contributes to the voltage across the electrodes while $\phi^{(0)}$ can be used for a complete description in the unelectroded region [28]. For a fully electroded crystal plate as shown in Fig. 1, $\phi^{(1)}$ is independent of the coordinates $x_1$ and $x_3$ in the plane of plate while $\phi^{(0)}=0$ in this case [28]. In order to investigate the effect of a strong electric field and consider more complicated structure such as a partially electroded crystal plate, we still need to retain all electric potentials in the expansion in (1).

The non-linear Green strain–mechanical displacement and electric field–electric potential relations are [24,28]

$$\begin{aligned} &S_{kl}=\frac{1}{2}(u_{k,l}+u_{l,k}+u_{m,k}u_{m,l}), \quad k,l,m=1,2,3,\\ &E_i=-\phi_{,i}, \quad i=1,2,3. \end{aligned} \tag{2}$$

Substitubting (1) into (2), we have the non-linear strain–displacement and electric field–electric potential relations as

$$\begin{aligned} &S_{kl}=\sum_{n=0}^{\infty} S_{kl}^{(n)}x_2^n,\\ &E_r=-\phi_{,r}^{(0)}-\frac{x_2}{2b}\phi_{,r}^{(1)}-\left(\frac{x_2^2}{b^2}-1\right)\phi_{,r}^{(2)}-\frac{x_2}{b}\left(\frac{x_2^2}{b^2}-1\right)\phi_{,r}^{(3)}, \quad r=1,3,\\ &E_2=-\frac{1}{2b}\phi^{(1)}-\frac{2x_2}{b^2}\phi^{(2)}-\left(\frac{3x_2^2}{b^3}-\frac{1}{b}\right)\phi^{(3)}, \end{aligned} \tag{3}$$

with

$$\begin{aligned} S_{kl}^{(n)}=&\frac{1}{2}[u_{k,l}^{(n)}+u_{l,k}^{(n)}+(n+1)(\delta_{2l}u_k^{(n+1)}+\delta_{2k}u_l^{(n+1)})]\\ &+\frac{1}{2}\left\{\sum_{s=0}^{n}[u_{m,l}^{(s)}+\delta_{2l}(s+1)u_m^{(s+1)}]\cdot[u_{m,k}^{(n-s)}+\delta_{2k}(t+1)u_m^{(n-s+1)}]\right\}, \end{aligned} \tag{4}$$

where $\delta_{2i}$ is the Kronecker delta [5].

Now the non-linear constitutive relations with Cauchy stress and Lagrangean strain tensors are [9,28]

$$\begin{aligned} &T_{ij}=c_{ijkl}S_{kl}+\frac{1}{2}c_{ijklmn}S_{kl}S_{mn}-e_{kij}E_k-\frac{1}{2}e_{klij}E_kE_l-\frac{1}{6}e_{klmij}E_kE_lE_m,\\ &D_i=e_{ikl}S_{kl}+\varepsilon_{ik}E_k+\frac{1}{2}\varepsilon_{kji}E_kE_j+\frac{1}{6}\varepsilon_{kjli}E_kE_jE_l, \quad i,j,m,n,k,l=1,2,3, \end{aligned} \tag{5}$$

where $T_{ij}$, $D_i$, $c_{ijkl}$, $e_{kij}$, and $\varepsilon_{ik}$ are stress tensor, electric displacement vector, elastic constants, piezoelectric constants, and dielectric constants, respectively. Generally, $c_{ijklmn}$, $e_{klij}$, $e_{klmij}$, $\varepsilon_{kji}$, and $\varepsilon_{kjli}$ are non-linear material constants which include the effect of the Maxwell electrostatic stress tensor [26,28]. We found that the linear portions of the non-linear constitutive equations in (5) are identical with the linear piezoelectric constitutive equations of the Mindlin plate theory [5]. The Cauchy stress and Lagrangean strain tensors are used in the formulation for simplicity of equations analogous to the linear equations.

The variational equation of non-linear stress equations of motion and electrostatics is [9,10,28]

$$\begin{aligned} &\int_V\{[(T_{ij}+T_{ik}u_{j,k})_{,i}-\rho\ddot{u}_j]\delta u_j+(D_{r,r}+D_{2,2})\delta\phi\}\,dV=0,\\ &i,j,k=1,2,3, \quad r=1,3, \end{aligned} \tag{6}$$

where the integral is over the volume of a plate as shown in Fig. 1, and $\rho$ is the density of a material. For a finite crystal plate with