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Scalar generalization of Newtonian restitution for simultaneous impact

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ABSTRACT

Simultaneous multiple impacts of solid bodies are modeled approximately within rigid body mechanics using impulse momentum relations, friction inequalities, and some kind of restitution model. The common restitution models are Newtonian, Poisson and energetic restitution. Of these, there is so far no satisfactory generalization of Newtonian restitution to simultaneous multiple impacts in three dimensions with friction. Here we propose a new generalization of Newtonian restitution which imposes a single scalar inequality regardless of the number of contacts. The inequality is a weighted sum of restitution inequalities for the different contacts, based on their respective coefficients of restitution. The weights used involve something we call local normal inertias, defined via the impulse momentum relations at each contact. The new generalized Newtonian restitution model, coupled with physical constraints of nonnegative normal impulses, non-interpenetration, and contact-wise friction inequalities, defines a feasible set of impulses. Our proposed impact model chooses the energy minimizing impulse within this feasible set, found using quadratic programming. Kinetic energy increases are never predicted by our model. Additionally, the model makes physically more realistic predictions than the popular linear complementarity approach in several cases, as we show using examples. For completeness, all relevant system matrices are described, and Matlab code is provided.

1. Introduction

Rigid body dynamics is useful in many areas including vehicle dynamics, robotics, granular mechanics, and simulations for games and movies. Collisions or impacts between rigid bodies require additional constitutive assumptions, called impact models, for their resolution. Since the collisional contact forces between real bodies are determined by deformations (however small), accurate prediction of general real-body impact outcomes is unlikely within rigid body mechanics. Nevertheless, for approximate descriptions that may suffice for some applications, rigid body impact models have been studied by many authors. For a representative sample of the literature, see [1–8], as well as the books by [9–11].

Rigid body impact models assume the following. The interaction is brief and involves large contact forces. Changes in configuration, and the effects of bounded non-contact forces, are negligible. So are the "$\boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega}$" terms of rigid body mechanics. Contact interactions occur through equal and opposite impulses at contact points. The bodies do not interpenetrate. The normal component of the impulse at each contact location is non-tensile. At each contact location, the magnitude of the tangential impulse is less than or equal to some coefficient μ times the normal impulse.[1] Finally, the net kinetic energy dissipation in the collision is nonnegative. The assumptions listed in this paragraph may be viewed as *fundamental physical restrictions*. Note that these restrictions do not uniquely determine the impact outcome, for which additional modeling assumptions must be made.

This paper is about models for simultaneous multiple impacts. Examples of simultaneous impacts are plentiful: a marble tossed into a bowl full of marbles, a rod pivoting on a ground contact to hit a wall, a ball dropped on another ball on a floor, and so on. In particular, this paper presents a new generalization of Newtonian restitution for multiple impacts. The key idea is as follows. All impact models we know of assume that with n_c simultaneous contacts, there should be n_c restitution conditions. We suggest that even with $n_c > 1$ contacts, one can have just *one* restitution condition for the entire impact interaction. That restitution condition,

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[1] This friction model is less fundamental than the other assumptions, but is widely adopted.

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which is a generalization of Newtonian restitution, leads to a simultaneous impact model that makes more accurate predictions for several impacts than presently available comparable models. The background information needed is presented in the rest of this introduction, and the model along with examples presented in subsequent sections of this paper.

Most models for simultaneous multiple impacts fall into two categories. One category models the impact interaction using evolution equations (e.g., [1,12]); important in this category are the so-called soft contact models, and sometimes other incremental equations are written as well.[2] The second category uses some kind of algebraic mapping from the pre-impact to post-impact state (e.g., [13,14]); these are sometimes referred to as hard contact models.

There are trade-offs in accuracy and complexity between algebraic and incremental models. Algebraic models tend to be computationally faster, and require no detailed information about relative compliances at different contacts; but they are generally less accurate in their predictions. Incremental models tend to involve greater computation and more detailed information, but can be more accurate under suitable circumstances. Both categories have their respective areas of utility. Our contribution in this paper is to algebraic models.

We note that algebraic laws for simultaneous impacts can be of three main types. The first treats the multiple-impact as a sequence of pairwise impacts (see [15–17]). This approach allows the use of Newtonian restitution and can give appealing results, like separation of the last ball in Newton's cradle. However, one may encounter long sequences of pairwise impacts, and indeterminacy in the choice of such sequences. The second type of algebraic law uses some form of the linear complementarity problem (LCP), and possibly a split of the total impact into a compression and a restitution phase with an added assumption of Poisson restitution. For frictionless collisions these complementarity-based laws become essentially identical, and lead to one key physical prediction: separation at initially touching locations *must* be accompanied by zero impulse at those locations.[3] These models therefore cannot predict separation of the last ball in Newton's cradle. The third, and so far least popular, type of algebraic law is based on energy minimization using quadratic programming. The only example we know of is [27], which studies maximally inelastic or most-dissipative impacts. Note that the most dissipative impact of a ball dropped on the ground has zero restitution (no rebound at all).

Normal restitution in rigid body impacts has been defined in three main ways. First, as a ratio of normal components of velocities (classical kinematic or Newtonian restitution); second, as a ratio of normal components of impulses (classical Poisson restitution, as mentioned above); and third, in terms of recovery of strain energy in normal-direction compliance [11]. Of these, the latter two are popular in incremental approaches. A discrete two-stage modification of Poisson restitution has been used in linear complementarity based formulations. Newtonian restitution has not, so far, been generalized to multiple impacts that are treated simultaneously (as opposed to sequentially pairwise). We will present such a generalization in this paper.

With the above introduction, we now present the key new idea of our paper.

For single point impacts where the normal component of initial relative velocity at the contact point is $V_{i,n} < 0$ and the final (post-impact) value of the same is $V_{f,n} \geq 0$, the Newtonian restitution model is simply

$$V_{f,n} = -eV_{i,n},$$

where e is the coefficient of restitution. Usually, we assume that $0 \leq e \leq 1$. If there are n_c contacts, then all rigid body impact models we know of assume that n_c restitution conditions are needed. We propose here that even with $n_c > 1$ contacts, we can still use a single condition. We will explain that that condition must involve inequalities, and propose a weighted sum of restitution inequalities from all contact locations. We will suggest that that the weights on these inequalities should involve something we call local normal inertias at the corresponding contact locations. The resulting net inequality will put a lower bound on an overall measure of restitution in the simultaneous impact. This lower bound, coupled with energy minimization (which tends to reduce rebound levels), will complete the impact model. In the special case of $n_c = 1$, we will have the usual Newtonian restitution. For arbitrary impact configurations with multiple contacts, our model will obey all the fundamental physical restrictions mentioned above, and also make more accurate predictions than the LCP based approach in many cases.

We can thus summarize the scope of this paper in two lines as follows. We present a novel generalization of Newtonian restitution for simultaneous multiple impacts that prescribes a single scalar inequality governing a distributed version of restitution. This new restitution model, combined with energy minimization, gives a new, computationally efficient, simultaneous impact model that is more accurate than LCP based approaches at least for several well-known examples.

2. Our impact model

The algebraic collision map is built in several steps. Some of this section is standard rigid body mechanics and definition of notation, stated briefly for completeness. Key new aspects of our model appear in items 6 and 7 below:

1. *Impulse−momentum relations*: Let the contact points be numbered $1, 2, \ldots, n_c$. At each contact, equal and opposite impulses act on the bodies. Let contact j involve bodies k and m, with $k < m$. Let the contact impulse acting there on body m be a vector $P^{(j)}$, with $-P^{(j)}$ acting on body k. There will be n_c such impulses. Similarly, let the velocity of the contact point on body m, relative to the contact point on body k, be $V^{(j)}$. These $P^{(j)}$ s and $V^{(j)}$ s, with three components each, can be arranged in $(3n_c) \times 1$ column matrices called P and V respectively. By rigid body impulse momentum relations, the relation between P and the impact-induced change in V is *linear*. We expect equations of the form
$$\Delta V = \mathbf{W}P. \tag{1}$$
W will generally be symmetric and positive semidefinite. A method for obtaining **W** is given in Appendix A.
2. *Local coordinate systems*: A local coordinate system is chosen for each contact, such that the first element of $P^{(j)}$ denotes its normal component (directed from the lower index into the higher index body, i.e., from body k into body m if $k < m$). The second and third components of $P^{(j)}$ are taken to be along locally tangential, mutually perpendicular directions. In planar systems, it is simpler if one tangential direction is in the plane of the dynamics. Such choice of tangential directions, though not essential, simplifies item 4 below.
We will refer to the pre-collision column matrix of relative velocity vector components as V_i, and the post-collision version of the same as V_f. Both are $3n_c \times 1$ column matrices. As

[2] A specialized group of such soft contact models, for example, studies pulse propagation in granular media [23–26].

[3] See e.g., [18–21,13]; but also [22] for a critique.