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Stationary response of strongly non-linear oscillator with fractional derivative damping under bounded noise excitation

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ABSTRACT

The stochastic averaging method for strongly non-linear oscillators with lightly fractional derivative damping of order α $(0<\alpha\leq 1)$ subject to bounded noise excitations is proposed by using the generalized harmonic function. The system state is approximated by a two-dimensional time-homogeneous diffusion Markov process of amplitude and phase difference using the proposed stochastic averaging method. The approximate stationary probability density of response is obtained by solving the reduced Fokker–Planck–Kolmogorov (FPK) equation using the finite difference method and successive over relaxation method. A Duffing oscillator is taken as an example to show the application and validity of the method. In the case of primary resonance, the stochastic jump of the Duffing oscillator with fractional derivative damping and its P-bifurcation as the system parameters change are examined for the first time using the stationary probability density of amplitude.

1. Introductions

In the past decades, there have been numerous investigations on fractional calculus and its applications in the fields of engineering, science and economics [1,2]. Using fractional calculus, the frequency-dependent damping behavior of various materials can be described very well [3–5]. In particular, the response of non-linear oscillators with small fractional derivative damping has been studied by many authors using various methods [6–13].

Bounded noise is a harmonic function with constant amplitude and random frequency and phase. It was firstly proposed by Stratonovich [14], and is a good model for many random excitations in engineering. For example, by adjusting its parameters, the Dryden and von Karman spectra of wind turbulence can be fit very well [15]. It can be a reasonable model for traveling loads and structures [16,17]. So far, the behavior of a variety of systems under bounded noise excitations has been studied, including the stochastic response [18–20], stability [18,21,22], chaotic motion [23–25], jump and bifurcation [26,27] of strongly non-linear oscillators. In addition, the principal parametric resonance, external resonance and internal resonances of some systems under bounded noise excitation have been systematically studied [28,29]. The dynamic behavior of non-linear system subject to bounded noise excitations is more complicated compared with that of the same system under broadband noise excitations. So far, to the authors' knowledge, the response of fractionally damped stochastic systems subject to bounded noise excitations and effect of fractional order on stochastic jump and bifurcation have not been studied yet.

In the present paper, a stochastic averaging method for strongly non-linear oscillators with lightly fractional derivative damping under bounded noise excitations is developed. The method is applied to study the response of Duffing oscillator with fractional derivative damping under bounded noise excitation. The reduced averaged FPK equation is solved using the finite difference method and successive over relaxation method. Based on the stationary probability density of amplitude, the stochastic jump and its bifurcation of Duffing oscillator with fractional derivative damping as the fractional order changes are investigated. The analytical solutions are verified using digital simulation.

2. Stochastic averaging

The stochastic averaging method for strongly non-linear oscillators with small fractional derivative damping under external and/or parametric excitations of Gaussian white noise [13], and combined harmonic and white noise [30] have been developed. In the present paper, the method is extended to the case of strongly non-linear system with fractional derivative damping under bounded noise excitation.

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Consider a strongly non-linear oscillator with small fractional derivative damping under bounded noise excitation. The equation of motion is of the form

$$\ddot{X}(t)+\varepsilon D^{\alpha}X(t)+g(X)=\varepsilon f(X,\dot{X})\xi(t) \tag{1}$$

where X is displacement; '·' denotes the derivative with respect to time t; ε is a small positive parameter; $\varepsilon D^{\alpha}X(t)$ denotes small fractional derivative damping; $g(X)$ represents a strongly non-linear restoring force; $\varepsilon f(X,\dot{X})$ denotes the amplitude of excitation; and $\xi(t)$ is a bounded noise. There are many definitions for fractional derivatives [1]. Herein, the following Riemann–Liouville definition is adopted:

$$D^{\alpha}X(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{X(t-\tau)}{\tau^{\alpha}}d\tau,\quad 0<\alpha\le 1 \tag{2}$$

where $\Gamma(\bullet)$ is a gamma function.

The bounded noise is of the form

$$\xi(t)=\cos(\Omega t+\sigma B(t)+\chi). \tag{3}$$

where Ω and σ^2 are constants representing center frequency and strength of frequency perturbation, respectively. $B(t)$ is standard Wiener process and χ is random phase uniformly distributed in [0, 2π]. The spectral density of $\xi(t)$ can be found as

$$S(\omega)=\frac{\sigma^2}{4\pi}\frac{\omega^2+\Omega^2+\sigma^4/4}{(\omega^2-\Omega^2-\sigma^4/4)^2+\sigma^4\omega^2} \tag{4}$$

Its auto correlation function is

$$R(\tau)=\frac{1}{2}\exp\left(-\frac{\sigma^2}{2}|\tau|\right)\cos\Omega\tau \tag{5}$$

and its probability density is

$$p(\xi)=\frac{1}{\pi\sqrt{1-\xi^2}} \tag{6}$$

It is obviously that $\xi(t)$ is non-Gaussian. The bandwidth of process $\xi(t)$ depends mainly on parameter σ. It is a narrow-band process when σ is small and a wide-band process when σ is large. As σ approaches infinite, the bounded noise becomes a white noise with constant spectral density. Conversely, as σ approaches zero, the bounded noise becomes a sinusoidal function with random phase.

The non-linear conservative oscillator associated with system (1) is

$$\ddot{x}(t)+g(x)=0 \tag{7}$$

Assume that Eq. (7) has a family of periodic solutions in domain U around the origin of the phase plane $(x,\dot{x})$. The periodic solutions can be expressed as

$$x(t)=a\cos\theta(t) \tag{8}$$

$$\dot{x}(t)=-a\upsilon(a,\theta)\sin\theta(t) \tag{9}$$

where

$$\theta(t)=\varphi(t)+\gamma \tag{10}$$

$$\upsilon(a,\theta)=\frac{d\varphi}{dt}=\sqrt{\frac{2[V(a)-V(a\cos\theta)]}{a^2\sin^2\theta}}. \tag{11}$$

here a is constant determined by potential energy

$$V(x)=\int_0^x g(u)du \tag{12}$$

and total energy

$$H=\frac{1}{2}\dot{x}^2+V(x) \tag{13}$$

as follows:

$$V(a)=V(-a)=H \tag{14}$$

$\cos\theta(t)$ and $\sin\theta(t)$ are the so-called generalized harmonic functions. a, υ (a,θ), and γ denote the amplitude, instantaneous frequency and initial phase angle, respectively, of the oscillator.

Expand υ into Fourier series

$$\upsilon(a,\theta)=C_0(a)+\sum_{n=1}^{\infty}C_n(a)\cos n\theta \tag{15}$$

Integrating Eq. (15) with respect to θ from 0 to 2π leads to the following approximate averaged frequency:

$$\omega(a)=\frac{1}{2\pi}\int_0^{2\pi}\upsilon(a,\theta)d\theta=C_0(a) \tag{16}$$

of the oscillator. Then, θ (t) in (10) can be approximated as

$$\theta(t)\approx\omega(a)t+\gamma \tag{17}$$

when ε is small, the solution to Eq. (1) is assumed of the following form:

$$Q(t)=X(t)=A\cos\Theta(t)$$

$$P(t)=\dot{X}(t)=-A\upsilon(A,\Theta)\sin\Theta(t) \tag{18}$$

where

$$\Theta(t)=\Phi(t)+\Gamma(t) \tag{19}$$

$$\upsilon(A,\Theta)=\frac{d\Phi}{dt}=\sqrt{\frac{2[V(A)-V(A\cos\Phi)]}{A^2\sin^2\Phi}} \tag{20}$$

in which A, Θ, Φ, and Γ are all random processes, A is related to H in a similar way as Eq. (14). Treating Eq. (18) as a generalized van der Pol transformation from $X,\dot{X}$ to A, Γ, one can obtain the following equations for the amplitude A and the phase angle Γ:

$$\frac{dA}{dt}=\varepsilon[F_{11}(A,\Theta)+F_{12}(A,\Theta,\Omega t+\Lambda)]$$
$$\frac{d\Gamma}{dt}=\varepsilon[F_{21}(A,\Theta)+F_{22}(A,\Theta,\Omega t+\Lambda)] \tag{21}$$

where

$$\Lambda=\sigma B(t)+\chi$$
$$F_{11}=\frac{A\upsilon\sin\Theta}{g(A)}D^{\alpha}(A\cos\Theta)$$
$$F_{21}=\frac{\upsilon\cos\Theta}{g(A)}D^{\alpha}(A\cos\Theta)$$
$$F_{12}=-\frac{A\upsilon\sin\Theta}{g(A)}f(A\cos\Theta,-A\upsilon\sin\Theta)\cos(\Omega t+\Lambda)$$
$$F_{22}=-\frac{-\upsilon\cos\Theta}{g(A)}f(A\cos\Theta,-A\upsilon\sin\Theta)\cos(\Omega t+\Lambda). \tag{22}$$

For narrow-band excitation, resonant case is more interesting. Assumed that σ is small and

$$\frac{\Omega}{\omega(A)}=\frac{q}{p}+\varepsilon\delta \tag{23}$$

where p and q are relatively prime small positive integers and δ is a detuning parameter.

Multiplying Eq. (23) by t and using approximate relations (17) and (19), we obtain:

$$\Omega t=\frac{q}{p}\Theta+\varepsilon\delta\Phi-\frac{q}{p}\Gamma \tag{24}$$

Introduce a new variable

$$\Delta=\varepsilon\delta\Phi-\frac{q}{p}\Gamma+\Lambda \tag{25}$$