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Global and bifurcation analysis of a structure with cyclic symmetry

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ABSTRACT

This article combines the application of a global analysis approach and the more classical continuation, bifurcation and stability analysis approach of a cyclic symmetric system. A solid disc with four blades, linearly coupled, but with an intrinsic non-linear cubic stiffness is at stake. Dynamic equations are turned into a set of non-linear algebraic equations using the harmonic balance method. Then periodic solutions are sought using a recursive application of a global analysis method for various pulsation values. This exhibits disconnected branches in both the free undamped case (non-linear normal modes, NNMs) and in a forced case which shows the link between NNMs and forced response. For each case, a full bifurcation diagram is provided and commented using tools devoted to continuation, bifurcation and stability analysis.

1. Introduction

In this paper, both free and forced vibrations of a non-linear cyclic symmetric structure are studied. The structure is composed of n_b identical substructures which undergo large strains. This system is typical when one studies bladed disks [2,26]. After modeling the system, a set of coupled non-linear differential equations in which non-linearity appears by cubic terms is obtained.

In the linear case, the study of the linear normal modes (LNMs) reveals a majority of double eigenfrequencies, corresponding to distinct eigenforms [22]. These eigenforms are associated with nodal diameter vibration modes. As these LNMs arise from an eigenvalue problem, there are always as many LNMs as degrees of freedom (dofs). Moreover, in the free or forced case, only one solution exists for a given frequency.

In the case of non-linear systems with cyclic symmetry, it has been shown that the number of non-linear normal modes (NNMs [7]) can exceed the number of dofs, the extra NNMs being generated through bifurcations or internal resonances [7,27]. Localized non-linear modes are an example of this property; they correspond to a free motion in which only a few substructures vibrate with non-negligible amplitude and they have no counterpart in the linear theory [2,29]. In the forced case, these additional NNMs give rise to a number of additional resonances leading to multiple solutions as in [29] where Vakakis showed a very complicated structure of resonance for a non-linear system with cyclic symmetry by using the multiple scales method. He showed that for a given frequency many solutions can coexist, some of them being stable.

Not only can multiple solutions coexist but also can they be disconnected from each other. In this latter case, classical methods based on continuation and bifurcation analysis fail at finding the disconnected branches of solutions. If one wants to dimension a structure properly – by considering all the possible solutions – one then needs to adopt a global analysis (GA) approach. Several solutions are mentioned in the literature (see [23] for an overview) with a common drawback even for small systems which is the computation cost. The GA method proposed in this paper takes advantage of the cubic form of the non-linearity combined with a reformulation of equations through the harmonic balance method (HBM); the resulting systems in the free undamped case for searching NNMs as well as in a forced case can then be solved (globally) in a reasonable amount of time.

Section 2 describes the system and its dynamical and HBM equations. The global analysis principle is then explained in Section 3.1; this is followed by recalls on continuation methods, bifurcation and stability analysis in Section 3.2. Finally these methods are applied to the undamped free system in order to find NNMs in Section 4 and to a forced case which exhibits a very rich response in Section 5.

2. Studied system

2.1. General description and dynamical equations

The studied structure has a cyclic symmetry property and can therefore be broken up into n_b identical sectors (Fig. 1). Each

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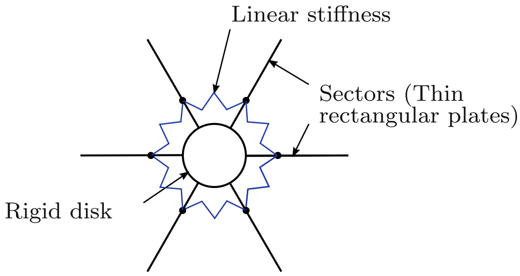

Fig. 1. Bladed wheel model used for establishing equations.

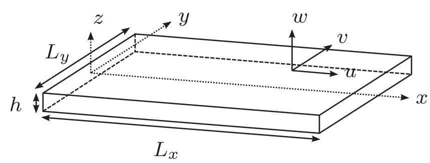

Fig. 2. Diagram of a rectangular plate and corresponding coordinate system.

sector is modeled by a thin rectangular plate clamped to the rigid disk which itself is fixed (Fig. 2). Consecutive plates are coupled by a linear stiffness while non-linearity is introduced by taking into account their large deflection. Plane stress assumption and the Love–Kirchhoff hypothesis (cross-sections exhibit solid body motion and remain perpendicular to the deformed surface of the middle sheet) are made. Then, plate displacements are entirely parameterized by their middle sheet transverse displacement w_j, $1 \le j \le n_b$. Moreover the material is assumed to follow a standard bi-dimensional Hooke's law leading to the following expression for the strain energy of plate j:

$$\begin{aligned} U_j = \frac{1}{2}\frac{Eh}{(1-\nu^2)}\int_{x=0}^{L_x}\int_{y=-L_y/2}^{L_y/2} & \left[\left(\frac{1}{2}\left(\frac{\partial w_j}{\partial x}\right)^2\right)^2 + \left(\frac{1}{2}\left(\frac{\partial w_j}{\partial y}\right)^2\right)^2 \right. \\ & \left. +2\nu\left(\frac{1}{2}\left(\frac{\partial w_j}{\partial x}\right)^2\frac{1}{2}\left(\frac{\partial w_j}{\partial y}\right)^2\right) + \frac{1-\nu}{2}\left(\frac{\partial w_j}{\partial x}\frac{\partial w_j}{\partial y}\right)^2\right] dx\,dy \\ & + \frac{1}{2}\frac{Eh^3}{12(1-\nu^2)}\int_{x=0}^{L_x}\int_{y=-L_y/2}^{L_y/2}\left[\left(\frac{\partial^2 w_j}{\partial x^2}\right)^2\right. \\ & \left. + \left(\frac{\partial^2 w_j}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w_j}{\partial x^2}\frac{\partial^2 w_j}{\partial y^2}\right) + \frac{1-\nu}{2}\left(2\frac{\partial^2 w_j}{\partial x \partial y}\right)^2\right] dx\,dy \end{aligned} \quad (1)$$

where E is Young's modulus and ν is Poisson's ratio.

The energy V^j of the linear stiffness localized at $(x_r,y_r) = (L_x/4,0)$ between plates j and $j+1$ is given by

$$V_j = \tfrac{1}{2}k(w_j(x_r,y_r) - w_{j+1}(x_r,y_r))^2 \quad (2)$$

for $1 \le j \le n_b$ with convention $j+1=1$ if $j=n_b$.

By neglecting rotary inertia the kinetic energy T_j of plate j is given by

$$T_j = \frac{1}{2}\rho h \int_{x=0}^{L_x}\int_{y=-L_y/2}^{L_y/2} \dot{w}_j^2 \, dx\,dy \quad (3)$$

In this paper, only a harmonic force, orthogonal to the plate, localized in $(x_f,y_f) = (L_x,0)$ is considered. The work W_j due to such an excitation on plate j is given by

$$W_j = w_j(L_x,0)\mathbf{F}_{ej} \quad (4)$$

The total energies U, T, V and W are then given by the sum over the number of plates n_b of the different local energies U_j, T_j, V_j and W_j. Equations of motions are finally derived by using Lagrange's equations along with a Rayleigh–Ritz approximation [16].

The remainder of this article will be devoted to the case of a system composed of $n_b = 4$ identical sectors, whose displacements are interpolated by a single Ritz function approximating the first bending mode thus yielding a non-linear problem with $n=4$ dofs that will serve as an example for the present work. The Ritz function is $\Phi(x,y) = (x/L_x)^2$ (consistent with the clamping at $x=0$) and leads to the following interpolation for transverse displacement of the j-th blade:

$$w_j(x,y) = q_j\Phi(x,y) \quad \text{for } 1 \le j \le n_b \quad (5)$$

By applying Lagrange's equations, and by adding a damping term, the following motion equations are obtained:

$$[\mathbf{M}]\ddot{\mathbf{X}} + [\mathbf{C}]\dot{\mathbf{X}} + [\mathbf{K}]\mathbf{X} + \beta\mathbf{X}^3 = \mathbf{F}_e(t) \quad (6)$$

with the notations $\mathbf{X} = (q_j)_{1 \le j \le n}$ and $\mathbf{X}^3 = (q_j^3)_{1 \le j \le n}$. The vector $\mathbf{F}_e = \mathbf{F}_{e0}\cos(\omega t)$ stands for the external forces amplitude, $[\mathbf{M}] = [\mathbf{I}]$ is the mass matrix, $[\mathbf{C}] = \delta[\mathbf{I}]$ is the damping matrix, $[\mathbf{K}]$ is the stiffness matrix given by

$$[\mathbf{K}] = \begin{bmatrix} \alpha+2c & -c & 0 & -c \\ -c & \alpha+2c & -c & 0 \\ 0 & -c & \alpha+2c & -c \\ -c & 0 & -c & \alpha+2c \end{bmatrix} \quad (7)$$

and β is the non-linear stiffness coefficient. The definitions of α, β, c, and δ can be found in Appendix A along with their numerical values.

2.2. Harmonic balance method

The harmonic balance method (HBM) is widely used for the study of non-linear systems. Numerous applications can be found in the literature, showing its ability to treat strongly non-linear systems like friction between blades and casing [11,10] or geometric non-linearities [14,21]. One major advantage of the method is that it requires no assumption about the non-linearities' magnitudes and uses the same procedure for strongly and weakly non-linear models.

The HBM consists of a decomposition of the solution $\mathbf{X}$ in a truncated Fourier series:

$$\mathbf{X}(t) = \mathbf{A}_0 + \sum_{k=1}^{N_h} \mathbf{A}_k\cos(k\omega t) + \mathbf{B}_k\sin(k\omega t) \quad (8)$$

Injecting this development (8) in Eq. (6), and by projecting equations on the $[1,(\cos(k\omega t),\sin(k\omega t))_{1 \le k \le N_h}]$ basis using the following scalar product:

$$\langle f,g \rangle = \int_0^{2\pi/\omega} f(t)g(t)\,dt \quad (9)$$

one gets a system of $\tilde{n} = n \times (2N_h+1)$ non-linear algebraic equations with $n \times (2N_h+1)+1$ unknowns $\mathbf{A}_k$, $\mathbf{B}_k$ and ω.

The number of harmonics retained N_h is a very important parameter. Generally, the higher N_h is, the better the solution. However, in the case where the number of harmonics selected is high, the solution procedure can quickly become difficult and time consuming. Fortunately, in most cases the series converges fast enough and leads to systems with reasonable dimensions. In this article only the first harmonic is going to be retained ($N_h = 1$) and the constant term $\mathbf{A}_0$ is dropped due to the symmetry of the system. Depending on whether the system is free or forced, different formulations can be obtained.

Forced case: The equation to be solved in the forced case is given by (6). The solution $\mathbf{X}$ is sought in the following form:

$$\mathbf{X}(t) = \mathbf{A}\cos(\omega t) + \mathbf{B}\sin(\omega t) \quad (10)$$

which leads to the following set of $2n$ algebraic equations:

$$\mathbf{H}_l\tilde{\mathbf{X}} + \mathbf{H}_{nl}(\tilde{\mathbf{X}}) = \mathbf{H}_e \quad (11)$$