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A unified solution for vibration analysis of functionally graded cylindrical, conical shells and annular plates with general boundary conditions

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ABSTRACT

In this paper, a unified solution method for free vibration analysis of functionally graded cylindrical, conical shells and annular plates with general boundary conditions is presented by using the first-order shear deformation theory and Rayleigh–Ritz procedure. The material properties of the structures are assumed to change continuously in the thickness direction according to the general four-parameter power-law distributions in terms of volume fractions of constituents. Each of displacements and rotations of those structures, regardless of boundary conditions, is expressed as a modified Fourier series, which is constructed as the linear superposition of a standard Fourier cosine series supplemented with auxiliary polynomial functions introduced to eliminate all the relevant discontinuities with the displacement and its derivatives at the edges and accelerate the convergence of series representations. The excellent accuracy and reliability of the current solutions are confirmed by comparing the present results with those available in the literatures, and numerous new results for functionally graded cylindrical, conical shells and annular plates with elastic boundary conditions are presented. The effects of boundary conditions and the material power-law distribution are also illustrated.

1. Introduction

Functionally graded materials (FGMs) have a smooth and continuous variation of the material properties in the thickness direction. Due to the unique properties, the application of FGMs has been successfully extended to various fields. As for other shape kinds, conical, cylindrical shells and annular plates are very common structural elements. Recently, those structures made of FGMs have been utilized in various engineering fields, such as aircraft, space vehicles and military industries, and in some case they have to carry dynamic loads. Therefore, a good understanding of their vibration characteristics is necessary for designers and users. The purpose of this paper is to study the dynamic behavior of those structures derived from shells of revolution.

FGM shells vibration problems deal with two main concepts: shell theories and computational approaches. In the last decades, a huge amount of research efforts have been devoted to vibration analysis and dynamic behaviors of the shells and a lager variety of shell theories and computational methods have been proposed and developed by researchers.

As far as the shell deformation theories in previous studies are concerned, there are a significant number of theories, including two-dimensional (2-D) theory, three-dimensional (3-D) elasticity theory. Most commonly used 2-D theories can be classified into three main categories: the classical shell theory (CST), the first-order shear deformation theory (FSDT), and the higher-order shear deformation theory (HSDT). More detailed descriptions on the development of researches on this subject may be found in several monographs respectively by Leissa [1], Qatu [2], Reddy [3], and Carrera [4]. Many researchers employed the CST to analyze various characteristics of thin shell structures [5–16]. Since the effect of shear deformation through the thickness is ignored, the CST is only suitable for thin shell structures and gives proper results at low frequencies. In order to eliminate the deficiency of the CST, the FSDT was developed, which assumes constant states of the transverse shear stresses though the shell thickness. There exist a large number of studies regarding shell structures based on FSDTs [17–34]. The shear correction factor, which is regarded as a constant in the calculation, is introduced to adjust the transverse shear stiffness. However, the value of the shear correction factor varies with material properties and boundary conditions in fact. The limitations of the FSDT necessitate the development of the HSDTs in which no such coefficients are required and the effects of both shear and normal deformations are considered. A number of HSDTs were developed [35–41]. Compared with 2-D theories,

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since the 3-D elasticity theory does not rely on any hypotheses, such theory not only provides realistic results but also brings out physical insights. A few investigations were carried out based on 3-D elasticity theory [42–47]. Although the HSDTs and 3-D elasticity theory can provide more accurate solutions, they introduce mathematical and computational complexities and require more computational demanding compared with those FSDTs. From the existing literature, we can know that the first-order shear deformation theory with proper shear correction factor is adequate for the prediction of the vibration behaviors of moderately thick shells. Therefore, in the present work, the first-order shear deformation shell theory is just employed to formulate the theoretical model.

The computational modeling of FGMs is an important tool to understanding of the structural behaviors, and has been the target of intensive research. A huge amount of research efforts have been devoted to the vibration analysis of FGM cylindrical shells in the literature and a number of analytical and numerical methods have been proposed and developed [9–12,21–25,38–40,42,43]. Loy et al. [9] used the Ritz method to study the influence of the constituent volume fractions and the effects of the boundary conditions on the vibration frequencies. Haddadpour [11] performed free vibration analysis of simply supported FGMs cylindrical shells for four sets of in-plane boundary conditions by Galerkin′s method. Iqbal et al. [12] employed the wave propagation approach to study the vibration characteristics of graded material circular cylindrical shells. Zhao et al. [21] investigated the static response and free vibration of FGM shells subjected to mechanical or thermo-mechanical loading based on Sander′s first order shear deformation shell theory by using the element-free *kp*-Ritz method. Tornabene et al. [23,24] used generalized differential quadrature (GDQ) method to analyze the dynamic behavior of FGM conical, cylindrical shells and annular plates. A general formulation for solving the free, steady-state and transient vibrations of FGM shells subjected to arbitrary boundary conditions is presented Qu et al. [25] by means of a modified variational principle on the basis of the first-order shear deformation shell theory. Due to the mechanical complexity of the structures, there are a few but not many publications related to the vibration analysis of FGM conical shells besides the aforementioned works of Tornabene et al. [23,24] and Qu [25]. A few methods have been developed to analyze the dynamic behaviors of FGM conical shells [15,16,28,29,44]. Sofiyev [15,16] investigated the vibration and stability behavior of FGMs conical shells under external loads with free and clamped boundary conditions by using the Galerkin method. Zhao and Liew [29] employed the element-free *kp*-Ritz method to analyze the free vibration of FGM conical shell panels. Malekzadeh [44] presented a three-dimensional free vibration analysis of the FGM truncated conical shells subjected to thermal environment, and the differential quadrature (DQ) method as an efficient method numerical tool is adopted to solve the thermal and thermo-mechanical governing equations. Compared with the analysis of FGM cylindrical, conical shells, the literature on FGM annular plates is very limited [23,24,32–34,41,45–47]. Efraim and Eisenberger [33] used exact element method to analyze the vibration of variable thickness annular isotropic and FGM plates based on FSDT. Hosseini-Hashemi and co-authors [41] presented an exact closed-form solution for free vibration of circular and annular moderately thick FG plates. Based on three-dimensional theory, dynamic analysis of multi-directional FGM annular plates is investigated by Nie and Zhong [45] using the state space-based differential quadrature method. Dong [47] presented three-dimensional free vibration analysis of FGM annular plates using Chebyshev–Ritz method.

From the review of the literature, most of the previous studies regarding the FGM cylindrical, conical shells and annular plates are confined to the classical boundary conditions. However, there are many non-classical boundary conditions such as elastic boundaries, which are often encountered in practical engineering applications, and there is a considerable lack of corresponding research regarding elastic boundary conditions. This work aims to provide a unified, efficient and exact solution for free vibration analysis of functionally graded cylindrical, conical shells and annular plates with general boundary conditions. The present work can be considered as an extension of authors′ previous works on vibration analysis of isotropic thin shells [48] and composite laminated shells [49,50] with general elastic boundaries. In this paper, a unified solution method for the free vibrations of functionally graded cylindrical, conical shells and annular plates with general boundary conditions is presented. The first-order shear deformation theory is adopted to formulate the theoretical model. The material properties of the structures are assumed to change continuously in the thickness direction according to the general four-parameter power-law distributions in terms of volume fractions of constituents. Each of displacements and rotations of those structures, regardless of boundary conditions, is expressed as a modified Fourier series. Mathematically, such a series expansion is capable of representing any function including the exact solutions. Rayleigh–Ritz procedure is used to obtain the exact solution base on the energy functions of those structures. The accuracy and reliability of the current solutions are confirmed by comparing the present results with those available in the literature. The effects of boundary conditions and the material power-law distribution are also illustrated.

2. Theoretical formulations

2.1. Description of the model

The geometry of shells considered hereafter is a surface of revolution. A differential element of such shell with its coordinate system is illustrated in Fig. 1. The thickness of shell element is represented by h respectively. The coordinate system composed of coordinates x, θ, and z is located on the reference surface ($z=0$) of the shell. $R_x$and R_θ are the radii of curvature of the reference surface along x and θ axes, respectively. The general boundary conditions are represented as three of independent linear springs (k_u, k_v and k_w) and two sets of rotational springs (K_x,K_θ) placed at the ends, and different boundary conditions can be obtained by setting

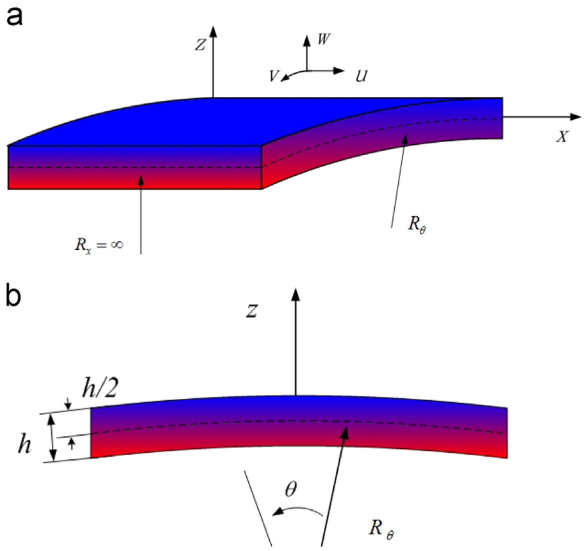

Fig. 1. Geometry and notations of differential element of structures.