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Green's functions for infinite and semi-infinite isotropic laminated plates

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ABSTRACT

The Green's functions for an infinite and a semi-infinite Kirchhoff isotropic laminated plate subjected to concentrated forces, concentrated moments, discontinuous displacements and slopes are obtained. The explicit expressions of the three 4×4 real matrices **H**, **L** and **S** for an isotropic laminated plate are derived by using the complex variable formulation recently developed by the authors. Once the Green's functions for an infinite plate are known, those for a semi-infinite plate can be conveniently obtained by using analytical continuation. The image forces on a point dislocation with discontinuous in-plane displacements and slopes due to its interaction with a rigidly clamped edge and a free edge are presented by using the obtained explicit expressions of **H** and **L**. Finally, the surface Green's functions of concentrated forces and moments are obtained as a limiting case of the Green's functions for a semi-infinite plate with a free edge. Some interesting features of the surface Green's functions are observed.

1. Introduction

The Green's function for Kirchhoff anisotropic laminated plates, in which the stretching and bending deformations are intrinsically coupled, has become an intensive research topic [1–8]. As pointed out by Cheng and Reddy [5] and Yin [7], the solutions derived by Becker [1] and Zakharov and Becker [2,3] are only valid for non-degenerate materials in which all the eigenvalues are distinct. Cheng and Reddy [4] derived real-form Green's function solutions for infinite and semi-infinite anisotropic plates by using the octet formalism [9]. However, it is not easy to obtain the 8×8 real matrices $\tilde{\mathbf{N}}$ and $\tilde{\mathbf{N}}(\theta)$ in the real-form solutions for a specific anisotropic laminate.

A Kirchhoff isotropic laminated plate is a mathematically degenerate material because there are only two independent eigenvectors associated with the quadruple roots $p_1 = p_2 = p_3 = p_4 = i$ [9]. Recently, Wang and Zhou [10] developed an elegant complex variable formulation for Kirchhoff isotropic laminated plates, which was originally derived by Beom and Earmme [11].

Even though the Green's functions for isotropic homogeneous plates with various complex configurations have been studied extensively (see e.g., [12–15]), those for isotropic inhomogeneous or laminated plates have rarely been addressed. The only available solution for a point force, a point moment and a dislocation in an infinite isotropic laminated plate was given by Beom and Earmme [11].

This paper aims to achieve two objectives. One is to derive the explicit expressions of the three 4×4 real matrices **H**, **L** and **S** for an isotropic laminated plate by using our recently developed complex variable formulation [10]. The original definition of the three real matrices can be found in [9]. The other is to obtain Green's functions for an infinite and a semi-infinite isotropic laminated plate subjected to concentrated forces, concentrated moments, discontinuous displacements and slopes. By using the existing results in the sextic and octet Stroh formalisms [9,16], and employing the explicit expressions of **H** and **L** derived in this work, the image forces on a point dislocation with discontinuous in-plane displacements and slopes due to its interaction with a rigidly clamped edge and a free edge will be presented. Through a limiting procedure, the surface Green's functions of concentrated forces and moments will be obtained and compared with the existing surface Green's function solutions for an isotropic homogeneous semi-infinite plate (e.g., the Flamant solution).

It should be pointed out that one of the main applications of the derived Green's functions is that they can be further used to find the Eshelby's tensor of isotropic laminated plates. Consequently, the overall mechanical properties of these laminated plates can be predicted. For more details, interested readers may refer to [17–19].

2. Formulation

This section is organized as follows. For readers' convenience, Section 2.1 will give a brief introduction of the fundamental complex variable formulation that we have developed in [10].

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Section 2.2 will then present how to derive the explicit expressions of the three real matrices **H**, **L** and **S** for an isotropic laminated plate by using such complex variable formulation.

2.1. Complex variable formulation

Consider a plate composed of an isotropic, linearly elastic material that can be inhomogeneous and laminated in the thickness direction. The plate has uniform thickness h, with the main plane of its undeformed form being located at $x_3=0$ in a Cartesian coordinate system $\{x_i\}(i=1,2,3)$. The displacement field in the Kirchhoff plate theory is given by

$$\tilde{u}_\alpha = u_\alpha + x_3\vartheta_\alpha, \quad \tilde{u}_3 = w, \tag{1}$$

where u_α, w, and $\vartheta_\alpha=-w_{,\alpha}$ represent the in-plane displacements, deflection, and slopes on the main plane, respectively and are all independent of x_3.

The membrane stress resultants $N_{\alpha\beta}$, bending moments $M_{\alpha\beta}$, transverse shearing forces R_β, and the modified Kirchhoff transverse shearing forces V_α that exclusively apply to free edge are defined by

$$N_{\alpha\beta}=Q\sigma_{\alpha\beta},\ M_{\alpha\beta}=Qx_3\sigma_{\alpha\beta},\ R_\beta=M_{\alpha\beta,\alpha},$$
$$V_1=R_1+M_{12,2},\ V_2=R_2+M_{21,1}, \tag{2}$$

where $Q(\cdots)=\int_{-h_0}^{h-h_0}(\cdots)dx_3$ with h_0 being the distance between the main plane and the lower surface of the plate.

When there is no external loading applied on the top and bottom surfaces of the plate, the equilibrium equations can be written as

$$N_{\alpha\beta,\beta}=0, \quad R_{\beta,\beta}=0, \tag{3}$$

which can be satisfied by introducing four stress functions φ_α and η_α such that [9]

$$N_{\alpha\beta}=-\in_{\beta\omega}\varphi_{\alpha,\omega},\ M_{\alpha\beta}=-\in_{\beta\omega}\eta_{\alpha,\omega}-\tfrac{1}{2}\in_{\alpha\beta}\eta_{\omega,\omega},$$
$$R_\alpha=-\tfrac{1}{2}\in_{\alpha\beta}\eta_{\omega,\omega\beta},\ V_\alpha=-\in_{\alpha\omega}\eta_{\omega,\omega\omega}, \tag{4}$$

where $\in_{\alpha\beta}$ are the components of the two-dimensional permutation tensor.

The constitutive equations for the laminated isotropic plate can be written as [11]

$$N_{\alpha\beta}=A_{\alpha\beta\omega\rho}\varepsilon_{\omega\rho}+B_{\alpha\beta\omega\rho}\kappa_{\omega\rho},$$
$$M_{\alpha\beta}=B_{\alpha\beta\omega\rho}\varepsilon_{\omega\rho}+D_{\alpha\beta\omega\rho}\kappa_{\omega\rho}, \tag{5}$$

where $\varepsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are the main plane strains and curvatures; $A_{\alpha\beta\omega\rho}, B_{\alpha\beta\omega\rho}$ and $D_{\alpha\beta\omega\rho}$ are the extensional, coupling and bending stiffness tensors given by

$$A_{\alpha\beta\omega\rho}=A_{12}\delta_{\alpha\beta}\delta_{\omega\rho}+\frac{1}{2}(A_{11}-A_{12})(\delta_{\alpha\omega}\delta_{\beta\rho}+\delta_{\alpha\rho}\delta_{\beta\omega}),$$
$$B_{\alpha\beta\omega\rho}=B_{12}\delta_{\alpha\beta}\delta_{\omega\rho}-\frac{1}{2}B_{12}(\delta_{\alpha\omega}\delta_{\beta\rho}+\delta_{\alpha\rho}\delta_{\beta\omega}),$$
$$D_{\alpha\beta\omega\rho}=D_{12}\delta_{\alpha\beta}\delta_{\omega\rho}+\frac{1}{2}(D_{11}-D_{12})(\delta_{\alpha\omega}\delta_{\beta\rho}+\delta_{\alpha\rho}\delta_{\beta\omega}). \tag{6}$$

In Eq. (6), $\delta_{\alpha\beta}$ is the Kronecker delta, $A_{ij}=QC_{ij}$, $B_{ij}=Qx_3C_{ij}$, and $D_{ij}=Qx_3^2C_{ij}$ $(ij=11,12)$, where $C_{11}=E/(1-\nu^2)$ and $C_{12}=\nu E/(1-\nu^2)$ with E and ν being the Young's modulus and Poisson's ratio of the plate, respectively. We choose the main plane to make $B_{11}=0$, which results in that $h_0=\int_0^h X_3C_{11}dX_3/\int_0^h C_{11}dX_3$ with $X_3=x_3+h_0$ representing the vertical coordinate of the given point from the lowest surface of the plate.

By substituting Eq. (5) into Eq. (3), we obtain

$$(A_{11}+A_{12})u_{\beta,\beta\alpha}+(A_{11}-A_{12})u_{\alpha,\beta\beta}=0,\ w_{,\alpha\alpha\beta\beta}=0, \tag{7}$$

which shows that stretching and bending are decoupled in the equilibrium equations. Consequently, the membrane stress resultants, bending moments, transverse shearing forces, in-plane displacements, deflection and slopes on the main plane of the plate, and the four stress functions can be concisely expressed in terms of four analytic functions $\phi(z)$, $\psi(z)$, $\Phi(z)$ and $\Psi(z)$ of the complex variable $z=x_1+ix_2$ as follows [9–11]:

$$N_{11}+N_{22}=4\text{Re}\{\phi'(z)+B\Phi'(z)\},$$
$$N_{22}-N_{11}+2iN_{12}=2[\bar{z}\phi''(z)+\psi'(z)+B\bar{z}\Phi''(z)+B\Psi'(z)],$$
$$M_{11}+M_{22}=4D(1+\nu^D)\text{Re}\{\Phi'(z)\}+\frac{B(\kappa^A-1)}{\mu}\text{Re}\{\phi'(z)\},$$
$$M_{22}-M_{11}+2iM_{12}=-2D(1-\nu^D)[\bar{z}\Phi''(z)+\Psi'(z)]-\frac{B}{\mu}[\bar{z}\phi''(z)+\psi'(z)],$$
$$R_1-iR_2=4D\Phi''(z)+\frac{B(\kappa^A+1)}{2\mu}\phi''(z), \tag{8}$$

$$2\mu(u_1+iu_2)=\kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)},$$
$$\vartheta_1+i\vartheta_2=\Phi(z)+z\overline{\Phi'(z)}+\overline{\psi(z)},\ w=-\text{Re}\{\bar{z}\Phi(z)+\chi(z)\},$$
$$\varphi_1+i\varphi_2=i[\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}]+iB[\Phi(z)+z\overline{\Phi'(z)}+\overline{\psi(z)}],$$
$$\eta_1+i\eta_2=iD(1-\nu^D)[\kappa^D\Phi(z)-z\overline{\Phi'(z)}-\overline{\psi(z)}]+i\frac{B}{2\mu}[\kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}], \tag{9}$$

where $\Psi(z)=\chi'(z)$, and

$$\mu=\frac{1}{2}(A_{11}-A_{12}),\ B=B_{12},\ D=D_{11},\ \nu^A=\frac{A_{12}}{A_{11}},\ \nu^D=\frac{D_{12}}{D_{11}},$$
$$\kappa^A=\frac{3A_{11}-A_{12}}{A_{11}+A_{12}}=\frac{3-\nu^A}{1+\nu^A},\ \kappa^D=\frac{3D_{11}+D_{12}}{D_{11}-D_{12}}=\frac{3+\nu^D}{1-\nu^D}. \tag{10}$$

2.2. Explicit expressions of *H*, *L* and *S*

It is stressed here that the octet formalism in [9] is still valid if x_3=0 in the Cartesian coordinate system is chosen on any plane parallel to the mid-plane of the plate. In the development of the octet formalism for Kirchhoff anisotropic plates, Cheng and Reddy [9] introduced three 4×4 real matrices **H**, **L** and **S**. Furthermore, Cheng and Reddy [4] gave an indirect proof that the two symmetric matrices **H** and **L** are positive definite, and they formally proved in [20] that the two matrices are positive definite. Below we derive the explicit expressions of **H**, **L** and **S** for isotropic laminated plates following the method described in Chapters 6.4 and 13.2 in [16]. Eq. (9) can be equivalently expressed into the following matrix forms:

$$\begin{bmatrix}u_1\\u_2\\\vartheta_1\\\vartheta_2\end{bmatrix}=\text{Re}\left\{\mathbf{A}'\begin{bmatrix}\bar{z}\phi'(z)+\psi(z)\\\phi(z)\\\bar{z}\Phi'(z)+\Psi(z)\\\Phi(z)\end{bmatrix}\right\},\ \begin{bmatrix}\varphi_1\\\varphi_2\\\eta_1\\\eta_2\end{bmatrix}=\text{Re}\left\{\mathbf{B}'\begin{bmatrix}\bar{z}\phi'(z)+\psi(z)\\\phi(z)\\\bar{z}\Phi'(z)+\Psi(z)\\\Phi(z)\end{bmatrix}\right\}, \tag{11}$$

where

$$\mathbf{A}'=\begin{bmatrix}-\frac{1}{2\mu} & \frac{\kappa^A}{2\mu} & 0 & 0\\ -\frac{i}{2\mu} & -\frac{i\kappa^A}{2\mu} & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & i & -i\end{bmatrix}, \tag{12}$$

$$\mathbf{B}'=\begin{bmatrix}-i & i & -iB & iB\\ 1 & 1 & B & B\\ \frac{iB}{2\mu} & \frac{iB\kappa^A}{2\mu} & iD(1-\nu^D) & iD(1-\nu^D)\kappa^D\\ -\frac{B}{2\mu} & \frac{B\kappa^A}{2\mu} & -D(1-\nu^D) & D(1-\nu^D)\kappa^D\end{bmatrix}. \tag{13}$$

Consequently, the impedance matrix $\mathbf{M}=-i\mathbf{B}'\mathbf{A}'^{-1}$ and its inverse $\mathbf{M}^{-1}=i\mathbf{A}'\mathbf{B}'^{-1}$ can be determined as