International Journal of Non-Linear Mechanics 45 (2010) 357–369

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Forces associated with non-linear non-holonomic constraint equations

Carlos M. Roithmayr [a,*], Dewey H. Hodges [b]

[a] Vehicle Analysis Branch, NASA Langley Research Center, Mail Stop 451, Hampton, VA 23681, USA
[b] School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA

A R T I C L E I N F O

Article history:
Received 6 March 2009
Received in revised form
15 October 2009
Accepted 19 December 2009

Keywords:
Kane's method
Constraint forces
Constraint torques
Lagrange multipliers
Undetermined multipliers
Non-holonomic constraint equations

A B S T R A C T

A concise method has been formulated for identifying a set of forces needed to constrain the behavior of a mechanical system, modeled as a set of particles and rigid bodies, when it is subject to motion constraints described by non-holonomic equations that are inherently non-linear in velocity. An expression in vector form is obtained for each force; a direction is determined, together with the point of application. This result is a consequence of expressing constraint equations in terms of dot products of vectors rather than in the usual way, which is entirely in terms of scalars and matrices. The constraint forces in vector form are used together with two new analytical approaches for deriving equations governing motion of a system subject to such constraints. If constraint forces are of interest they can be brought into evidence in explicit dynamical equations by employing the well-known non-holonomic partial velocities associated with Kane's method; if they are not of interest, equations can be formed instead with the aid of vectors introduced here as non-holonomic partial accelerations. When the analyst requires only the latter, smaller set of equations, they can be formed directly; it is not necessary to expend the labor first to form the former, larger set and subsequently perform matrix multiplications.

Published by Elsevier Ltd.

1. Introduction

Motion constraints imposed on a mechanical system are described with non-holonomic (non-integrable) constraint equations, whereas configuration constraints are expressed with holonomic constraint equations. Two examples of motion constraints with which the reader may be familiar are the condition of rolling, which is the absence of slipping, and the restriction on velocity imposed by a sharp-edged blade. These constraints are sometimes described with equations written in the matrix form $\alpha u+\beta=0$, where u is a column matrix of motion variables $u_1,\ldots,u_n$. Motion variables, also referred to as generalized speeds, are in general linear combinations of the time derivatives of generalized coordinates, $\dot{q}_1,\ldots,\dot{q}_n$. The distinguishing feature of such equations is that they are linear in the motion variables. However, one may consider motion constraints that must be described by relationships that are inherently non-linear in the motion variables, having the form $f(q_1,\ldots,q_n,u_1,\ldots,u_n,t)=0$. In Ref. [1] Bajodah et al. review some of the literature dealing with non-linear non-holonomic constraint equations and consider it important to study them because they can arise in connection with servo-constraints or program constraints when a control system enters the picture. As explained in Refs. [2,3], such constraints are enforced by application of control forces as opposed to the forces present when bodies and particles come into contact with one another, as is the case with classical, passive constraints.

Golubev states in Ref. [4] that, as of yet, there is no example of a passive mechanical device that can compel a motion constraint described by an equation that is non-linear in velocity. Roberson and Schwertassek note in Ref. [5] that all known motion constraints imposed on purely mechanical systems can be expressed with relationships that are linear in velocity variables. Unfortunately, the relationships in such situations are often artificially teased into non-linear forms to create contrived examples used to illustrate a proposed procedure. For instance, a non-linear equation is devised in Ref. [6] to describe the constraint imposed on a rolling disk. The well-known Appell–Hamel mechanism is studied and discussed, for example, in Refs. [1,7–12]. It is recognized in Refs. [1,8–12] that the constraints imposed on this mechanical system can be expressed with linear relationships, but despite this the mechanism is used in Refs. [1,11,12] to demonstrate the application of methods for dealing with non-linear non-holonomic constraint equations. In Refs. [13,14], Zekovich offers several examples of passive mechanical systems in which the constraints are described with non-linear non-holonomic constraint equations. In what follows it is shown that the associated constraints can in fact be expressed with linear non-holonomic equations. Another example studied in

* Corresponding author. Tel.: +1 757 864 6778.
E-mail addresses: c.m.roithmayr@larc.nasa.gov (C.M. Roithmayr), dhodges@gatech.edu (D.H. Hodges).

0020-7462/$ - see front matter Published by Elsevier Ltd.
doi:10.1016/j.ijnonlinmec.2009.12.009

Refs. [8,15–17] involves a device proposed by Benenti in Ref. [18]. However, a purely mechanical system is involved and therefore, according to the observations in Refs. [4,5], the non-linear non-holonomic equation used to describe the constraint must be regarded as contrived.

Whenever a motion constraint can be expressed entirely with linear non-holonomic constraint equations, it should be dealt with accordingly. Any number of approaches can be used to deal with the equations in their linear form; the exercise of cajoling such equations into a non-linear appearance serves no useful purpose. The new approaches contained in this paper, and the examples of their application, are concerned strictly with inherently non-linear non-holonomic constraint equations.

The literature contains several instances of motion constraints described by non-holonomic equations that are inherently non-linear in velocity. Perhaps the simplest case, provided by Golubev in Ref. [4], involves a single particle P that is subject to a uniform gravitational field and moves in a vertical plane fixed in an inertial reference frame N. The magnitude of the velocity ${}^{N}\mathbf{v}^{P}$ of P in N is to remain constant. The particle thus constrained serves as a model of a robot manipulator tip used to spray-paint a wall or polish a surface. Variations of this problem are studied in Refs. [19–21]. A familiar example proposed by Appell, in which P moves in three dimensions, is discussed in Refs. [7,15,19,22,23]. Special cases of Appell's problem are examined in Refs. [20,24]. Control of an inverted pendulum constitutes an example studied in Refs. [15,16]. A thin rigid rod moves in a vertical plane in the presence of a uniform gravitational field, with the lower end of the rod always in contact with a horizontal line. The system is referred to as Marle's servomechanism; as proposed in Ref. [7], an actuator controls the horizontal displacement of the rod's lower end according to some control law in order to keep the rod vertical. An earlier paper by Huston and Passerello [25] considers the more general case of balancing a pole whose lower end remains in contact with a horizontal plane, while the pole is otherwise free to move in the space above the horizontal plane.

The forthcoming developments in this paper are carried out for the most part in terms of vectors. These quantities are used also in expressing the main results, and discussing the contributions of the work. By *vector* we mean a basis-independent quantity having direction and magnitude, such as position, velocity, acceleration, or force, involved in the application of elementary principles of dynamics to study motion taking place in three-dimensional space. Other examples of a vector include partial velocities and partial angular velocities associated with advanced principles of dynamics. We do not mean a row or column matrix whose elements consist of three basis-dependent scalar measure numbers of a vector. Nor do we have in mind a matrix containing more than three scalar elements, such as a collection of generalized forces, or a row or column matrix considered from the viewpoint of linear algebra to belong to an n-dimensional tangent space, orthogonal space, etc.

In Ref. [26], a comprehensive, consistent, and concise method is established for identifying a set of forces needed to constrain the behavior of a mechanical system modeled as a set of particles and rigid bodies. The method is exercised in Ref. [27] with an example involving a configuration constraint, and a motion constraint expressed with an equation that is linear in velocity. The purpose of this paper is to apply the method to constraints described by non-holonomic equations that are inherently non-linear in velocity. (It is to be understood that the term "velocity," used in the general case of a system of particles, subsumes "angular velocity" in the special case in which a subset of particles makes up a rigid body. The term "acceleration" likewise encompasses an angular counterpart.) An essential feature of the method consists of expressing constraint equations in vector form rather than entirely in terms of scalars and matrices as is customary. A constraint equation that has been differentiated once or twice with respect to time, so that it contains the acceleration of a point or the angular acceleration of a rigid body, is said to be written at the acceleration level. Likewise, a constraint equation at the velocity level is one that has been differentiated at most once, so that it contains the velocity of a point or the angular velocity of a rigid body. It so happens that the method discussed in Refs. [26,27] can be applied whenever constraints can be described at the acceleration level by a set of independent equations that are linear in acceleration; therefore, it is applicable to constraint equations that are non-linear in velocity when written at the velocity level.

The method in question yields expressions in vector form for constraint forces, and for torques of couples formed by constraint forces (hereafter referred to as constraint forces and constraint torques). Thus, the directions of these vectors are identified, together with the specific point at which a constraint force must be applied, and the particular body upon which a constraint torque must be exerted. Such information about the vector quantities is of interest in its own right, and is to be preferred over the information contained in a matrix whose elements are scalar generalized constraint forces. In the process of constructing generalized constraint forces, information about the direction, magnitude, and point or body of application of constraint forces and torques becomes lost; in principle, each generalized constraint force is a sum of contributions from every constraint force and torque acting on a mechanical system. Although generalized constraint forces can be computed in a straightforward manner from knowledge of constraint forces and torques, usually it is impractical to invert the process and recover the original information about constraint forces and torques from generalized constraint forces.

Anderson is concerned in Ref. [28] with configuration constraints and with motion constraints described by non-holonomic equations that are linear in the motion variables. Although such constraints are not the direct subject of the present investigation, Anderson makes an observation that is nevertheless relevant to our discussion. Often, a Lagrange multiplier or undetermined multiplier used to treat a constrained system is not related in a clear way to any particular constraint force or torque. In the method introduced here, each multiplier has a straightforward relationship to a constraint force and/or torque.

The emphasis in this paper is on analytic derivation of equations of motion that do or do not contain evidence of forces and torques needed to impose motion constraints described with inherently non-linear non-holonomic equations. This stands in contrast to methods of computational dynamics, where the object is numerical formulation and solution of equations of motion. With knowledge of constraint forces and torques obtained by inspection of constraint equations written in vector form, and the two new approaches developed here, the analyst can form explicit equations of motion by hand or with the aid of symbolic algebra software. Equations that do not contain evidence of constraint forces can be formed directly; they need not be obtained from numerical manipulations of equations in which evidence of constraint forces is present.

The remainder of the paper is organized as follows. First, a treatment of non-linear non-holonomic constraint equations is undertaken in Section 2 for a generic system of particles; the results are applicable whether or not a subset of particles makes up a rigid body. The method of Ref. [26] is used to identify directions of constraint forces and the particles to which they must be applied. The constraint forces are used together with extensions to Kane's method [30] to obtain two new ways of deriving dynamical equations of motion. The first of these is