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On the uniqueness of large deflections of a uniform cantilever beam under a tip-concentrated rotational load

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ABSTRACT

The problem of a uniform cantilever beam under a tip-concentrated load, which rotates in relation with the tip-rotation of the beam, is studied in this paper. The formulation of the problem results in non-linear ordinary differential equations amenable to numerical integration. A relation is obtained for the applied tip-concentrated load in terms of the tip-angle of the beam. When the tip-concentrated load acts always normal to the undeformed axis of the beam (the rotation parameter, $\beta=0$) there is a possibility of obtaining non-unique solution for the applied load. This phenomenon is also observed for other rotation parameters less than unity. When the tip-concentrated load is acting normal to the deformed axis of the beam ($\beta=1$), many load parameters are obtained for a tip-angle with different deformed configurations of the beam. However, each load parameter corresponds to a tip-angle, which confirms the uniqueness on the solution of non-linear differential equations.

1. Introduction

Barten [1,2], Bisshopp and Drucker [3] and Frisch-Fay [4] have studied the non-linear bending of cantilever beam subjected to end loads. The problem has been solved in terms of elliptic integrals [3,5] as well as using different numerical techniques [6–9]. Rao and Rao [11] and Rao et al. [12] have examined the large deflection behavior of a cantilever beam subjected to a tip-concentrated load (P), which rotates ($\beta\phi(0)$) in relation with the tip-rotation, $\phi(0)$ of the beam shown in Fig. 1. In all these studies only one equilibrium shape was obtained for a beam with a prescribed tip-concentrated load. Wang [10] and Navaee and Elling [13] have studied the large deflection of cantilever beams subjected to inclined end loads. They have found that for each combination of the beam and loading condition, there are certain numbers of equilibrium configuration for the beam. Some interesting studies were made on the multiple equilibrium solutions of the uniform cantilever under a dead load (rotation parameter, β=0) [14–17] and analytical solutions for follower force (β=1) [18,19].

Recently, Shvartsman [20] has presented a direct method for the large deflection problem of a cantilever beam under a tip follower force (β=1). The results were found to be in good agreement with Rao and Rao [11] for β=1. This method fails to give the results of Rao and Rao [11] for $\beta\neq 1$. The method of Shvartsman [20] needs two times of integration for the specified load parameter. First time integration gives the tip-angle for the specified load parameter in addition to the slope ($\phi(s)$) of the deformed beam from the free end. Tip-coordinates of the beam are obtained by evaluating the integrals through the Simpson's rule. Specifying the tip-deflections and the tip-angle for the load parameter and integrating the resultant differential equations, the deformed configuration of the beam can be obtained. Shvartsman [20] presents only the tip-angle and the tip-deflections for the specified load parameter for which one time integration is sufficient. To obtain the deformed configuration of the beam accurately, it is essential to convert the integrals into differential equations and integrate the non-linear differential equations specifying the obtained tip-angle and tip-deflections for the load parameter. Otherwise, large amount of $\phi(s)$ data has to be stored for evaluation of the integrals to obtain the deformed configuration of the beam from the first integration. Further, the complexity of storing the data enhances with increase in the load parameter.

It is also noted from the numerical results [20] that there is a possibility of different load parameters for a specified tip-angle of a uniform cantilever beam. The problem of a uniform cantilever beam under tip-concentrated load, which rotates in relation with the tip-rotation of the beam is studied here. A general method is proposed for the rotation parameter $\beta\in[0,1]$. A relation is obtained for the load versus tip-angle of the beam to obtain the load parameter for a specified tip-angle. The deformed

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configurations of the beam are obtained by solving directly the resulting non-linear differential equations through fourth-order Runge–Kutta integration scheme.

2. Theoretical formulation

The formulation of the problem is mainly based on an important relation of the flexural theory (i.e.,$M/EI = 1/\rho = d\phi/ds$); the quantity $1/\rho$ (the curvature of the deflected axis of the beam) characterizes the magnitude of bending deformation, which is proportional to the bending moment, M and inversely proportional to the product EI, called the flexural rigidity of the beam.

The moment–curvature relationship of a uniform cantilever beam (see Fig. 1) subjected to a tip-concentrated rotational load (P) is as follows: [11]

$$EI\frac{d\phi}{ds} = P\cos\{\beta\phi(0)\}(X - X_a) + P\sin\{\beta\phi(0)\}(Y - Y_a) \tag{1}$$

where

$$X(s) = \int_s^L \cos\phi(\eta)d\eta \tag{2}$$

$$Y(s) = \int_s^L \sin\phi(\eta)d\eta \tag{3}$$

Here E is the Young's modulus, I is the moment of inertia, L is the length of the beam, η is a dummy variable, $\phi(0)$ is the tip-angle of the beam and β is rotation parameter, which lies between 0 and 1. At $s=0$, Eqs. (2) and (3) give tip-coordinates (X_a, Y_a) of the beam. The rotation parameter $\beta=0$ represents the problem with tip-concentrated load, which always acts normal to the undeformed axis of the beam, whereas $\beta=1$ represents the problem under a tip-concentrated load acting normal to the deformed axis of the beam.

Differentiating Eqs. (1)–(3) with respect to s, the following system of equations are obtained:

$$EI\frac{d^2\phi}{ds^2} + P\cos\{\beta\phi(0) - \phi\} = 0 \tag{4}$$

$$\frac{dX}{ds} = -\cos\phi \tag{5}$$

$$\frac{dY}{ds} = -\sin\phi \tag{6}$$

Boundary conditions for differential Eqs. (4)–(6) are

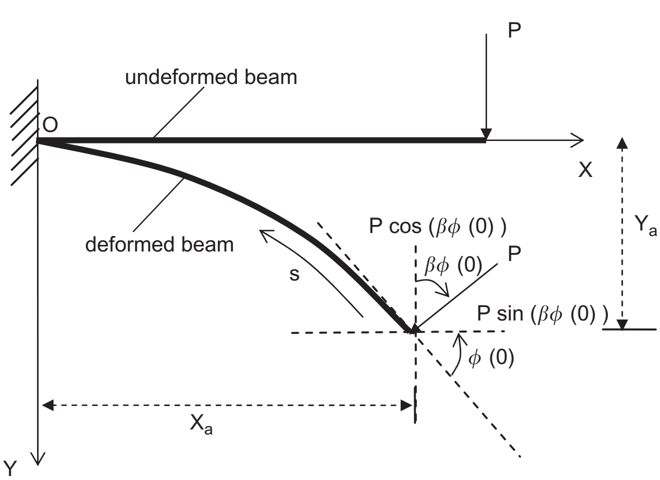

Fig. 1. A uniform cantilever beam under a tip-concentrated load (P), which rotates ($\beta\phi(0)$) in relation with the tip-rotation ($\phi(0)$) of the beam.

At the tip of the beam ($s=0$)

$$\frac{d\phi}{ds} = 0 \tag{7}$$

At the root of the beam ($s=L$)

$$\phi = 0,\ X = 0,\ Y = 0 \tag{8}$$

The solution of Eqs. (4)–(8) is obtained in terms of elliptic integrals for one equilibrium deformed configuration of the beam [11]. The problem governed by the second-order non-linear differential Eq. (4) with the conditions (7) and (8) is solved using the fourth-order Runge–Kutta integration scheme. Initially the two-point boundary value problem is converted to an initial value problem by estimating the tip-angle, $\phi(0)$ as one of the required initial condition for the specified load, in an iterative procedure, so as to satisfy the other boundary condition at the root of the beam (i.e., $\phi=0$ at $s=1$) [12]. The tip-coordinates (X_a,Y_a) of the beam are obtained by evaluating the integrals in Eqs. (2) and (3) using the Simpson's one-third rule. The deformed configuration of

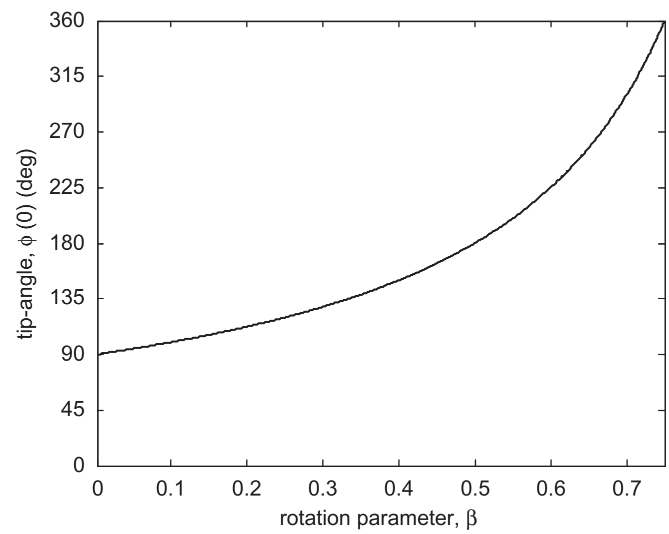

Fig. 2. Variation in tip-angle, $\phi(0)$ with the rotation parameter (β) at which the load parameter (λ) is undefined.

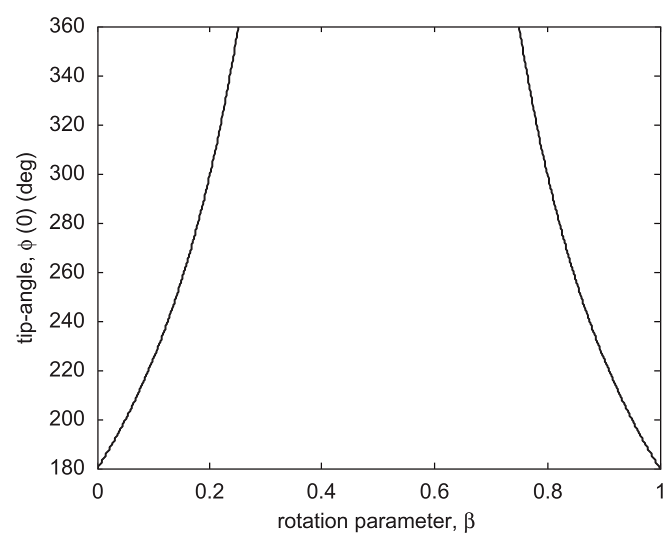

Fig. 3. Extreme value of tip-angle, $\phi(0)$ with the rotation parameter (β) beyond which the solution of the problem does not exist.