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The unsteady flow generated by an oscillating wall with transpiration

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A B S T R A C T

In this work, an analytical solution for the fluid behavior over flat plates with impulsive and oscillating motions, starting from rest, and with wall transpiration, is presented. The classical solution of this problem is given by Panton [7] and is found to be an especial case of the solution here presented. The analytical solution is obtained without the use of any special transformations, such as Laplace or Fourier transforms. Instead, an extension of the variable separation technique is used together with similarity arguments. A non-dimensional number—the transpiration rate—is used to take into account the injection or suction of fluid at the wall. This parameter is shown to be of great influence on the proposed velocity solution.

1. Introduction

The viscous flow a newtonian fluid can be described by the Navier–Stokes equations. These nonlinear partial differential equations form a very complex system, with a small number of exact solutions.

Although the exact solutions are limited to particular combinations of simple geometry and boundary conditions, they provide a great insight on more complex flow situations. In addition, exact solutions are very useful to assess the accuracy of approximate numerical and theoretical procedures as well as experimental practices.

One class of problems with known analytical solutions are the so-called Stokes problems. For a flat plate, these problems are related to the motion induced by an oscillating infinite plane wall in contact with a viscous fluid when the wall presents harmonic oscillations in the longitudinal direction. In first problem of Stokes [8], the wall is initially at rest and a transient flow is induced to the fluid by the suddenly application of an impulsive motion. In the second problem of Stokes, the motion is generated by an oscillating plate. In time, the transient motion vanishes and the fluid velocity at any point is just a harmonic oscillation with the same wall frequency. The latter problem was solved by Stokes [9]. The solution of the former problem, in closed form and in terms of tabulated functions, was given by Panton [7], who considered the solution to be a summation of transient and steady state parts. Erdogan [4] solved this problem through a Laplace transform technique. Using the same technique, Liu and Liu [6] proposed a solution for the extended Stokes problem, for a finite depth flow. Finally, Erdogan and Imrak [5] calculated the solution using the Fourier transform technique.

A more general solution for the Stokes problems can be derived when fluid transpiration at the wall is considered. The governing equations must then be modified with the addition of a new term representing the momentum introduced into the flow by the transpiration of fluid. This solution has significant application in boundary layer control with important examples in manufacturing techniques, aeronautical systems, chemical and mechanical engineering processes. The analysis to be developed here can also be applied to a more general problem—when the wall velocity is an arbitrary function—by using a Fourier series to represent the arbitrary condition and solving a sequence of Stokes problems.

In this paper, the analytical solution of the mentioned Stokes problems with addition of wall fluid injection or suction is presented. To the best of authors knowledge, this the first time that such a closed solution is presented. The solution shown here in the case of zero-transpiration rate reduces to the solution of Panton [7]. The proposed solution does not resort to a space transformation. Instead, a modified version of the variable separation technique is evoked. The final form uses the complementary complex error function. The solution when the plate has a impulsive start with wall transpiration is also presented and it turns out to be a specific case of Stokes transient solution.

In a recent article, Cruz and Pinho [2] solved the second Stokes problem for upper convected Maxwell (UCM) fluids. They used an approach similar to the one shown here, however, just the fully developed flow was analyzed.

The outline of the rest of this paper is as follows. In the second section, the basic equation for the Stokes transient and steady-state problems are shown. In the third section, the analytical solutions

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are developed. Next, some results are presented and physical interpretations are given. Finally, some conclusions are drawn.

2. Basic equations

The Stokes problem considered here is stated as follows: consider a fluid with viscosity ν, initially at rest, occupying a half plane $y \geq 0$ and bounded on the x-axis by an infinite plane wall. At time $t > 0$ the wall moves in x-direction with velocity given by $u_w(t)$. The fluid velocity $u \equiv u(y,t)$ is described by the Navier–Stokes equation, which can be cast as

$$\frac{\partial u}{\partial t} + V_w \frac{\partial u}{\partial y} - \nu \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } y>0 \text{ and } t>0 \tag{1}$$

where V_w is the transpiration velocity.

The boundary and initial conditions are

$$u = u_w(t) = u_0 \exp(i\omega t) \quad \text{at } y=0,\ t>0 \tag{2}$$

$$u = 0 \quad \text{at } y \to \infty,\ t>0 \tag{3}$$

$$u = 0 \quad \text{at } y=0,\ t=0 \tag{4}$$

where u_0 is the maximum amplitude of wall velocity oscillation, ω is the frequency of the wall velocity and $i=\sqrt{-1}$ is the imaginary constant. Using the wall velocity given in expression (2), the sine and cosine oscillations can be treated by taking the real and imaginary parts of the velocity field. Note that the cosine oscillation presents a discontinuity at t=0, when the wall velocity jumps from zero to u_0 differently from the sine function, which represents a more realistic situation. To keep the generality of solution presented here, the solution when a cosine oscillation is imposed on the flow is shown.

Consider the set of non-dimensional variables

$$U = \frac{u}{u_0}, \quad \tau = \omega t, \quad \eta = y\left(\frac{\omega}{\nu}\right)^{1/2}, \quad \xi = V_w/\sqrt{4\omega\nu} \tag{5}$$

which can be used to transform Eq. (1) according with

$$\frac{\partial U}{\partial \tau} + 2\xi \frac{\partial U}{\partial \eta} - \frac{\partial^2 U}{\partial \eta^2} = 0 \quad \text{for } \eta>0 \text{ and } \tau>0 \tag{6}$$

and the boundary conditions (2)–(4)

$$U = \exp(i\tau) \quad \text{at } \eta=0,\ \tau>0 \tag{7}$$

$$U = 0 \quad \text{at } \eta \to \infty,\ \tau>0 \tag{8}$$

$$U = 0 \quad \text{at } \eta=0,\ \tau=0 \tag{9}$$

Eq. (1) is the classical second problem of Stokes [8] with wall transpiration. The closed form solution to this set of equations, without transpiration, was given by Panton [7].

3. Solution technique

3.1. Periodic solution

The periodic solution is found after the start up phase effect dies out and the fluid experiments a harmonic motion with the same frequency of the wall. Further on, when we consider the start up phase, this solution will be useful. To the best of the present author's knowledge, this solution cannot be found in literature as fluid injection is present. A solution for this problem is developed in this section. The idea is to obtain an ordinary differential equation from Eq. (6), by taking a linear combination of the independent variables.

Consider that the horizontal velocity can be written as a function of $\varphi = A\tau + B\eta$ or $U = U(\varphi)$ and that A, B are two complex constants, so that

$$\frac{\partial U}{\partial \tau} = A\frac{dU}{d\varphi} \quad \frac{\partial U}{\partial \eta} = B\frac{dU}{d\varphi} \quad \text{and} \quad \frac{\partial^2 U}{\partial \eta^2} = B^2\frac{d^2U}{d\varphi^2} \tag{10}$$

Substitution of the above equations into Eq. (6), gives

$$\frac{A+2\xi B}{B^2}\frac{dU}{d\varphi} = B^2\frac{d^2U}{d\varphi^2} \tag{11}$$

The solution of this equation is

$$U(\varphi) = C_1\frac{B^2}{A+2\xi B}\exp\left(\frac{A+2\xi B}{B^2}\varphi\right) + C_2 \tag{12}$$

The constants can be determined through the boundary conditions

$$U(\tau, \eta=0) = U(\varphi = A\tau) = e^{i\tau} \tag{13}$$

$$U(\tau, \eta \to \infty) = U(\varphi = B\eta) = 0 \tag{14}$$

so that, $C_1 = (A+2\xi B)/B^2$, C_2 =0 and $A^2 + 2\xi BA - iB^2 = 0$.

A relation between constants A and B can then be immediately established,

$$B = i(-\xi \pm \sqrt{\xi^2 + i})A \tag{15}$$

The following simplification is obtained:

$$U(\tau,\eta) = \exp\left[i\left(\tau + \frac{B}{A}\eta\right)\right] \tag{16}$$

To satisfy the second boundary condition in Eq. (14), we must have

$$Re\left[i\frac{B}{A}\right] \leq 0 \quad \Rightarrow \quad Re[-\xi \pm \sqrt{\xi^2 + i}] \geq 0 \tag{17}$$

where $Re(z)$ is the real part of the complex argument z.

Parameter ξ can be either positive (injection) or negative (suction), but the term $\sqrt{\xi^2 + i}$ is strictly positive. The only way to maintain condition (17) is taking the positive sign at expression (15). Then, the final solution to the non-transient second problem of Stokes with transpiration can be written as

$$U = \exp[i\tau + \eta(\xi - \sqrt{\xi^2 + i})] \tag{18}$$

For the no-injection case, $\xi = 0$, we have

$$U = \exp\left(-\frac{\eta}{\sqrt{2}}\right)\left[\cos\left(\tau - \frac{\eta}{\sqrt{2}}\right) + i\sin\left(\tau - \frac{\eta}{\sqrt{2}}\right)\right] \tag{19}$$

Note that Eq. (19) is the exact solution to non-transient problem found by Stokes [9].

3.2. Start-up phase solution

Considering the horizontal velocity as a function $U = e^{\varphi}F(\tau,\eta)$, with $\varphi = A\tau + B\eta$, Eq. (6) can be cast as

$$\frac{\partial F}{\partial \tau} + 2(\xi - B)\frac{\partial F}{\partial \eta} + F(A + 2\xi B - B^2 t) = \frac{\partial^2 F}{\partial \eta^2} \tag{20}$$

Since A and B must satisfy $A + 2\xi B - B^2 = 0$ (see Eq. (15)), we have

$$\frac{\partial F}{\partial \tau} + 2(\xi - B)\frac{\partial F}{\partial \eta} = \frac{\partial^2 F}{\partial \eta^2} \tag{21}$$

Take $F \equiv F(\kappa)$ as a function of κ with the decomposition $\kappa = (\tilde{A}\tau + \tilde{B}\eta)g(\tau)$ where $\tilde{A}$, $\tilde{B}$ are two complex constants (there is no need to compute the values of these two constants since they cancel out in the calculations).

Substitution of κ into Eq. (21), gives

$$\frac{(\tilde{A}\tau + \tilde{B}\eta)g' + (\tilde{A} + 2\xi\tilde{B} - 2B\tilde{B})g}{\tilde{B}^2 g^2}\frac{dF}{d\kappa} = \frac{d^2F}{d\kappa^2} \tag{22}$$