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Stochastic responses of Duffing-Van der Pol vibro-impact system under additive and multiplicative random excitations☆

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ABSTRACT

The paper is devoted to an averaging approach to study the responses of Duffing-Van der Pol vibro-impact system excited by additive and multiplicative Gaussian noises. The response probability density functions (PDFs) are formulated analytically by the stochastic averaging method. Meanwhile, the results are validated numerically. In addition, stochastic bifurcations are also explored.

1. Introduction

Many non-smooth factors arise very naturally in engineering applications, such as impacts, collisions, dry frictions and so on. Most of the previous literatures fastened on the study of either non-smooth non-linear deterministic systems or non-smooth linear stochastic systems. For deterministic non-smooth systems, Shaw and Holmes [1] investigated the dynamics of a periodically forced impact system. Luo [2–5] and Xie [6] explored bifurcations and chaos of two-degree-of-freedom linear vibro-impact systems by the Pioncaré map. In recent years, the particular bifurcations unique to non-smooth systems have been examined extensively [7–12].

For stochastic non-smooth systems, Feng [13,14] explored the mean responses of a stochastic friction system and an impact system by introducing the mean Poincaré map. Huang and Zhu [15] obtained the stationary responses of a multi-degree-of-freedom vibro-impact system under white noise excitations according to the Hertz contact law. Zhuravlev [16] proposed a non-smooth variable transformation to deal with non-smooth characteristics. Based on the non-smooth variable transformation, recently, Dimentberg and Iourtchenko [17–22] studied the dynamics of linear stochastic vibro-impact systems, including impact energy losses, the response PDFs, subharmonic responses and stochastic chaotic responses. Namachchivaya [23,24] developed an averaging approach to study the dynamic behaviors of a vibro-impact system with random perturbations.

As is well known, a response PDF is an important characteristic of general non-linear stochastic systems, which appeals to either averaging approach [25,26] or numerical simulations. The theory of stochastic averaging approach has been proposed in Ref. [27]. In this paper, we will focus on the study of stationary response in a non-linear stochastic vibro-impact system. The paper is organized as follows. In Section 2, the transformed Duffing-Van der Pol system is derived by using the non-smooth transformation. In Section 3, stochastic averaging method is applied to deal with the transformed system and the approximate stationary response PDFs are derived. In Section 4, the directly numerical simulations verify the analytical results. Furthermore, stochastic bifurcations are also considered. Conclusions are presented in Section 5.

2. Problem statement

Consider the following Duffing-Van der Pol vibro-impact system under additive and multiplicative random excitations:

$$\ddot{x} + \omega_0^2 x + (c_2 x^2 - c_1)\dot{x} + c_3 x^3 = c_4 \xi_1(t) + c_5 x \xi_2(t), \quad x > 0, \tag{1a}$$

$$\dot{x}_+ = -r\dot{x}_-, \quad x = 0, \tag{1b}$$

where parameters $c_1, c_2, c_3, {c_4}^2, {c_5}^2$ are assumed of order $O(\varepsilon)$, and ω_0 is the natural frequency. r is the restitution coefficient which

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depicts impact losses and satisfies $0 < r \leqslant 1$. Specially, the impact is modeled as a classical elastic one when $r = 1$. $\xi_1(t)$ and $\xi_2(t)$ are standard Gauss white noises and independent correlation, namely

$$E[\xi_i] = 0, \quad E[\xi_i(t + \mathrm{d}t)\xi_j(t)] = \begin{cases} \delta(\mathrm{d}t), & i = j, \\ 0, & i \neq j, \end{cases} \quad (i, j = 1, 2).$$

Note that t_* is the instant of impacts, namely, $x(t_*) = 0$, which is not given in advance. $\dot{x}_+$ and $\dot{x}_-$ are the velocities of system after and before the instant of impacts t_*, respectively, that is $\dot{x}_\pm = \dot{x}(t_* \pm 0)$. Motion of system governed by Eq. (1a) is "unrestricted" until the constraint condition $x = 0$ is satisfied. The impact condition (1b) is then imposed.

Following Zhuravlev [15], the non-smooth transformation of state variables is introduced as follows:

$$x = x_1 = |y|, \quad \dot{x} = x_2 = \dot{y}\,\mathrm{sgn}\,y, \quad \ddot{x} = \ddot{y}\,\mathrm{sgn}\,y,$$
$$\mathrm{sgn}\,y = \begin{cases} 1, & y > 0, \\ -1, & y < 0. \end{cases} \tag{2}$$

Note that the transformation (2) maps the domain x > 0 of original phase plane $(x, \dot{x})$ onto the whole phase plane $(y, \dot{y})$. The transformed equation of motion can be written by substituting (2) into Eqs. (1) as

$$\ddot{y} + \omega_0^2 y + [c_2 y^2 - c_1]\dot{y} + c_3 y^3 = c_4 \mathrm{sgn}(y)\xi_1(t) + c_5 y \xi_2(t), \quad t \neq t_*, \tag{3a}$$

$$\dot{y}_+ = r\dot{y}_-, \quad t = t_*. \tag{3b}$$

Obviously, the transformed velocity $\dot{y}$ is continuous in the special case of elastic impact (i.e., $r = 1$) according to Eq. (3b). The jump of transformed velocity $\dot{y}$ becomes proportional to $1-r$ instead of $1+r$ for the jump of original velocity $\dot{x}$. Since $y(t_* \pm 0) = y(t_*) = 0$ and $\dot{y}_\pm = \dot{y}(t_* \pm 0)$, the Dirac delta function can be introduced $\delta(y) = \delta(y(t_*)) = \delta(t - t_*)/|\dot{y}(t_*)|$, which can be written as $\delta(t - t_*) = |\dot{y}|\delta(y)$. Further, combining with $\dot{y}(t)\delta(t - t_*) = \dot{y}(t_*)\delta(t - t_*)$, the impulsive term can be obtained:

$$(\dot{y}_- - \dot{y}_+)\delta(t - t_*) = (1 - r)\dot{y}|\dot{y}|\delta(y). \tag{4}$$

From Eqs. (3) and (4), the transformed equation can be made into

$$\ddot{y} + \omega_0^2 y + [c_2 y^2 - c_1]\dot{y} + c_3 y^3 - (r - 1)\dot{y}|\dot{y}|\delta(y) = c_4\,\mathrm{sgn}(y)\xi_1(t) + c_5 y \xi_2(t). \tag{5}$$

Thus, the original system (1) is reduced to Eq. (5). The term $(r - 1)\dot{y}|\dot{y}|\delta(y)$ on the right of Eq. (5) describes the impact losses of system, which can be regarded as an impulsive damping term.

3. Stochastic averaging approach

Under the foregoing assumption that damping, impact losses and excitation terms are small, the transformed Eq. (5) represents a system with a weakly non-linear restoring force, which permits the rigorous analytical study by stochastic averaging approach.

Letting $y = y_1$, $\dot{y} = y_2$, Eq. (5) is equivalent to a pair of first-order equations:

$$\dot{y}_1 = y_2, \tag{6a}$$

$$\dot{y}_2 = -\omega_0^2 y_1 - (c_2 y_1^2 - c_1)y_2 - c_3 y_1^3 - (1 - r)y_2|y_2|\delta(y_1) + c_4 \mathrm{sgn}(y_1)\xi_1(t) + c_5 y_1 \xi_2(t). \tag{6b}$$

Clearly, excitation terms are not associated with y_2, which implies Wong–Zaikai correction terms are zero. Eq. (6) can be converted to the Itô-type stochastic differential equations as follows:

$$\mathrm{d}y_1 = y_2\,\mathrm{d}t, \tag{7a}$$

$$\mathrm{d}y_2 = [-\omega_0^2 y_1 - (c_2 y_1^2 - c_1)y_2 - c_3 y_1^3 - (1 - r)y_2|y_2|\delta(y_1)]\,\mathrm{d}t + c_4\,\mathrm{sgn}(y_1)\mathrm{d}W_1(t) + c_5 y_1 \mathrm{d}W_2(t), \tag{7b}$$

where $W_1(t)$ and $W_2(t)$ are unit Wiener processes.

The corresponding total energy function and potential function are, respectively:

$$H(t) = \frac{y_2^2}{2} + \omega_0^2 \frac{y_1^2}{2} + c_3 \frac{y_1^4}{4}, \tag{8}$$

$$U(y_1) = \omega_0^2 \frac{y_1^2}{2} + c_3 \frac{y_1^4}{4}. \tag{9}$$

The derivative of Eq. (8) with respect to t can be obtained as follows:

$$\dot{H} = [-(c_2 y_1^2 - c_1)y_2 - (1 - r)y_2|y_2|\delta(y_1) + c_4 \mathrm{sgn}(y_1)\xi_1(t) + c_5 y_1 \xi_2(t)]y_2. \tag{10}$$

Then Eq. (10) can be converted to the Itô-type stochastic differential equation:

$$\mathrm{d}H = [-\omega_0^2 y_1 - (c_2 y_1^2 - c_1)y_2 - c_3 y_1^3 - (1 - r)y_2|y_2|\delta(y_1) + \tfrac{1}{2}c_4^2 + \tfrac{1}{2}c_5^2 y_1^2]\,\mathrm{d}t + [c_4 y_2 \mathrm{sgn}(y_1)]\,\mathrm{d}W_1(t) + [c_5 y_1 y_2 \mathrm{sgn}(y_1)]\,\mathrm{d}W_2(t). \tag{11}$$

As is well know, it is difficult to obtain the exact stationary solution to the corresponding Fokker–Planck–Kolmogorov (FPK) equation of Eq. (11) due to the non-linear nature.

Since coefficients c_1, c_2, $(1-r)$, $c_4{}^2$, $c_5{}^2$ are small, y_1 and y_2 are two fast varying random processes, while $H(t)$ is a slowly varying random process. Therefore, $H(t)$ may be approximated by a Markov process governed by the mean Itô-type stochastic differential equation:

$$\mathrm{d}H = m(H)\,\mathrm{d}t + \sigma(H)\,\mathrm{d}W(t), \tag{12}$$

where $W(t)$ is a unit Wiener process. $M(H)$ and $\sigma(H)$ are the mean drift coefficient and the mean diffusion coefficient, respectively, which can be derived by the quasi-conservative averaging procedure [24]. Note that the impulsive damping term should be averaged over a half-period since there are two impacts in each period.

The mean drift coefficient and the mean diffusion coefficient can be obtained, respectively:

$$m(H) = \frac{1}{T_{1/4}(H)}[c_1 R(H) - c_2 Q(H) + (r - 1)H + \frac{1}{2}c_4^2 T_{1/4}(H) + \frac{1}{2}c_5^2 P(H)], \tag{13}$$

$$\sigma^2(H) = \frac{1}{T_{1/4}(H)}[c_4^2 R(H) + c_5^2 Q(H)], \tag{14}$$

where

$$T_{1/4}(H) = \frac{T(H)}{4} = \frac{1}{\sqrt{E}}F_1(J),$$

$$R(H) = \frac{2\sqrt{E}}{3c_3}[(E - F)F_1(J) + (2F - E)F_2(J)],$$

$$Q(H) = \frac{4\sqrt{E}}{15c_3^2}[(3EF - 2E^2 - F^2)F_1(J) + (2E^2 + 2F^2 - 2EF)F_2(J)],$$

$$P(H) = \frac{2}{c_3\sqrt{E}}[(F - E)F_1(J) + EF_2(J)],$$

$$E = \omega_0^2 + c_3 A^2, \quad F = \frac{c_3 A^2}{2}, \quad J = \sqrt{\frac{F}{E}}. \tag{15}$$

The amplitude A is the positive root of the following equation:

$$U(A) = H, \quad \text{that is } A = U^{-1}(H).$$