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Weakly non-linear analysis of Rayleigh–Benard convection with time periodic heating

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ABSTRACT

The stability of a horizontal layer of fluid heated from below as well as from above is examined. The temperature gradient between the walls of the fluid layer consists of a steady part and a time-dependent part, which is oscillatory. The amplitude of temperature modulation is δ. By considering the weakly non-linear analysis, it is shown that the modulation produces a range of stable hexagons near the critical Rayleigh number. Some comparisons have been made with the other theoretical results.

1. Introduction

This paper deals with the stability of a fluid layer confined between two horizontal planes and heated from below and above periodically with time. Considerable attention has been given to this stability problem during the last 50 years. Convection in a horizontal layer of fluid is often accompanied by a cellular pattern of motions [1–3]. Stuart [4] discussed in detail the non-linear mechanics of hydrodynamic stability. Malkus and Veronis [5] have investigated the theory, by examining the stability of various solutions of the non-linear equations corresponding to stationary solutions. Later on Davis and Segel [6] studied the non-linear stability problem for free surfaces by considering surface deflection. They found the influence of the variable fluid properties on the cellular convection.

Davis [7] has suggested that weakly non-linear evolution at Rayleigh numbers near critical is governed by amplitude equation of Landau type. Finucane and Kelly [8] performed experiment on the alteration of the convective heat transport due to temperature modulation. Employing the shape assumption and free boundary conditions, they developed a non-linear analysis and estimated the two-dimensional convection amplitude. Davis et al. [9] have presented a weakly non-linear convective instability theory for pattern selection in single-component systems coupling Benard convection and solidification.

Schluter et al. [10] in their analysis concluded that in the absence of modulation, buoyancy driven convection is supercritical and has the form of two-dimensional rolls when the fluid has constant properties. But in case of modulation it has been shown by Roppo et al. [11] that the convection is transcritical and that hexagonal cells are the preferred stable mode in the neighbourhood of the critical Rayleigh number. Roppo et al. [11] considered temperature modulation only of the lower wall, using sinusoidal profile.

The object of the present investigation is to study the weakly non-linear convection by considering the temperature modulation of both the boundaries. Furthermore, the results have been obtained for more general temperature profiles. Here we consider a temperature profile, which is similar to the variation of the atmospheric temperature near the earth's surface during one complete day–night cycle. The temperature profile has been expressed in Fourier series. The results have been obtained for the following three cases: (a) when the plate temperatures are modulated in phase, (b) when the modulation is out of phase and (c) when only the lower plate temperature is modulated. The results have been compared with the other theoretical results. The results have their relevance with convective flows in the terrestrial atmosphere.

2. Statement of the problem

Consider a viscous, incompressible fluid layer, confined between two parallel, horizontal stress-free planes, a distance d apart.

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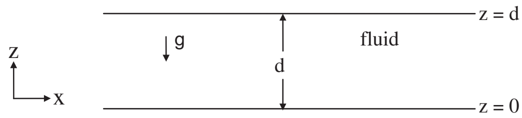

Fig. 1. Benard configuration.

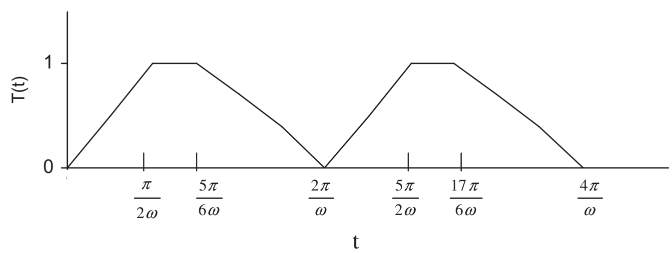

Fig. 2. Variation of the temperature T with time t.

The system is of infinite extent in the horizontal direction. The planes considered here are perfect heat conducting boundaries. The configuration is shown in Fig. 1.

The non-dimensional governing equations under the Boussinesq approximations are

$$\frac{1}{P}\left(\frac{\partial \mathbf{V}}{\partial t}+R^{1/2}\mathbf{V}\cdot\nabla\mathbf{V}\right)=\nabla^2\mathbf{V}-\nabla p+R^{1/2}\theta\hat{k}, \tag{1}$$

$$\nabla\cdot\mathbf{V}=0, \tag{2}$$

$$\frac{\partial\theta}{\partial t}+R^{1/2}\frac{\partial T_0}{\partial z}w+R^{1/2}\mathbf{V}\cdot\nabla\theta=\nabla^2\theta \tag{3}$$

where $P=v/\kappa$ is the Prandtl number and $R=\alpha g\Delta T d^3/v\kappa$ is the Rayleigh number, perturbation quantities $\mathbf{V}=(u,v,w)$, θ and p are, respectively, the fluid velocity, temperature and pressure fields. T_0 is the temperature in the conducting state while t is the time. α is the coefficient of volume expansion, g is the acceleration due to gravity, ΔT is the temperature difference between the walls, v is the kinematic viscosity, κ is the thermal diffusivity, $\hat{k}$ is the vertical unit vector.

To modulate the wall temperatures, we consider the temperature profile as shown in Fig. 2, known as day–night profile. This temperature profile is defined below:

$$T(t)=\begin{cases}\dfrac{2\omega t}{\pi} & 0\leqslant t\leqslant\dfrac{\pi}{2\omega}\\ 1 & \dfrac{\pi}{2\omega}\leqslant t\leqslant\dfrac{5\pi}{6\omega}\\ \dfrac{12}{7}\left(1-\dfrac{\omega t}{2\pi}\right) & \dfrac{5\pi}{6\omega}\leqslant t\leqslant\dfrac{2\pi}{\omega}\end{cases} \tag{4}$$

where ω is the modulation frequency and $2\pi/\omega$ is the period of oscillation. The temperature profile shown in Fig. 2 is similar to the variation of the earth's surface temperature during two complete day–night cycles.

The Fourier series of the function (4) is given by

$$T(t)=\frac{a_0}{2}+\sum_{m=1}^{\infty}a_m\cos m\omega t+\sum_{m=1}^{\infty}b_m\sin m\omega t \tag{5}$$

where

$$a_0=\tfrac{14}{12}, \tag{5a}$$

$$a_m=\frac{2}{m^2\pi^2}\left[\frac{-10}{7}+\cos\frac{m\pi}{2}+\frac{3}{7}\cos\frac{5m\pi}{6}\right], \tag{5b}$$

$$b_m=\frac{2}{m^2\pi^2}\left[\sin\frac{m\pi}{2}+\frac{3}{7}\sin\frac{5m\pi}{6}\right]. \tag{5c}$$

By shifting the origin we write

$$T(t)=\sum_{m=1}^{\infty}a_m\cos m\omega t+\sum_{m=1}^{\infty}b_m\sin m\omega t. \tag{6}$$

Now we define the externally imposed wall temperatures as follows:

(i) When the temperature of the lower boundary as well as of the upper boundary is modulated, we have

$$T(t)=T_R+\beta d\left[1+\frac{\delta}{2}\left\{\sum_{m=1}^{\infty}a_m\cos m\omega t+\sum_{m=1}^{\infty}b_m\sin m\omega t\right\}\right]\quad\text{at } z=0 \tag{7a}$$

$$T_R+\beta d\frac{\delta}{2}\left[\sum_{m=1}^{\infty}a_m\cos(m\omega t+\phi)+\sum_{m=1}^{\infty}b_m\sin(m\omega t+\phi)\right]\quad\text{at } z=d \tag{7b}$$

(ii) When the upper boundary is held at fixed constant temperature, then

$$T(t)=T_R+\beta d\left[1+\delta\left\{\sum_{m=1}^{\infty}a_m\cos m\omega t+\sum_{m=1}^{\infty}b_m\sin m\omega t\right\}\right]\quad\text{at } z=0 \tag{8a}$$

$$=T_R\quad\text{at } z=d. \tag{8b}$$

Here δ represents small amplitude, β is the thermal gradient, ϕ is phase angle and T_R is the reference temperature. The basic temperature gradient $\partial T_0/\partial z$, which appears in Eq. (3) is obtained in the dimensionless form as given below:

$$\frac{\partial T_0}{\partial z}=-1-\delta\left[Re\left\{\sum_{m=1}^{\infty}a_m f(\lambda_m)e^{im\omega t}\right\}+\mathrm{Im}\left\{\sum_{m=1}^{\infty}b_m f(\lambda_m)e^{im\omega t}\right\}\right] \tag{9}$$

where

$$f(\lambda_m)=\frac{\lambda_m}{\sinh\lambda_m}[\cos\lambda_m(1-z)-e^{i\phi}\cosh\lambda_m(z)]. \tag{9a}$$

and

$$\lambda_m^2=\mathrm{im}\,\omega^*\quad\text{and}$$
$$\omega^*=\omega d^2/\kappa\,(\text{non-dimensional frequency}). \tag{9b}$$

Henceforth, the asterisk will be dropped and ω will be considered as the non-dimensional frequency. Free–free boundary conditions are being applied in this problem, therefore at $z=0$ and 1, we have

$$w=0,\quad\frac{\partial^2 w}{\partial z^2}=0 \tag{10a}$$

and also

$$\theta=0. \tag{10b}$$

Since the boundaries are perfectly heat conducting, the fluctuation of temperature vanishes there.

The non-linear stability problem for the system (1)–(3) is posed in terms of a small parameter $\in$, which is a measure of amplitude of