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Exact fundamental thermo-elastic solutions of a transversely isotropic elastic medium with a half infinite plane crack

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ABSTRACT

This paper presents three-dimensional (3D) exact fundamental solutions of the thermoelastic field in a transversely isotropic elastic medium weakened by a half infinite plane crack subjected to a pair of point thermal loadings symmetrically acting on the crack surface. In view of the symmetric condition, the original problem is transformed into a mixed boundary value problem of a half space. By means of the general solutions conjugated with the generalized potential theory method, the problem is exactly solved and the corresponding Green's functions are derived, *for the first time*. Complete and exact fundamental solutions are expressed in terms of elementary functions. The singular behavior of the crack tip is discussed and the stress intensity factor is given explicitly.

1. Introduction

As an important branch in solid mechanics, the study of thermoelastic problems has long been prevailing in the literature; for example, wave propagation [1,2], inclusion [3], contact problems [4–6], dislocation [7], heat transfer [8], and so on. In particular, the thermoelastic crack problem subjected to various types of external thermal loadings has been discussed extensively in the scientific community [9–17]. Most of the previous analytical works treated the axisymmetric [9–13] or two dimensional cases [14,15]. In the past crack analyses, integral transform method [10,12,13] or theory of dual/triple integral equations [11,16,17] were usually adopted. It is noted that there exist quite few reports concerning the non-axisymmetric problems within the framework of thermo-elasticity in literature.

Kellogg [18] pointed that the potential theory method is a tool for studying problems governed by Laplace equations in several physic areas. This method is extended by Fabrikant [19,20] who creatively represented the reciprocal of the distance between two points in Euclidean space by an integral. Such a representation of the distance lends to various non-classic 3D elastic solutions for mixed boundary value problems arising in crack and punch problems, by using the potential theory along with the general solutions proposed by Elliott [21] for pure elasticity. The potential theory was further generalized by Chen and his coauthors for crack and contact problems in multi-coupling disciplines [22], for instance, piezo-elasticity [23], thermo-elasticity [24], thermo-piezo-elasticity [25], magneto-electro-thermo-elasticity [26], thermo-poro-elasticity [27], to name a few. In particular, the penny-shaped crack problem in Ref. [27] is axisymmetric, since the crack is subjected to pairs of identical mechanical, thermal and pressure loadings which are uniformly distributed on the upper and lower crack surfaces.

From the mathematical point of view, the structures of governing equations for crack problems in piezoelasticity and magnetoelectroelasticity are identical to their counterparts in pure elasticity [24]. However, this is not so when the thermal effect is taken into count; certain new features appear, making some challenges to the potential theory method [24,28]. It is noted that most works (Refs. [22–27], among others) associated with crack problems mentioned above were dedicated to penny-shaped cracks.

Half infinite crack has attracted a lot of attention from numerous scholars [29–34]. It is interesting to note that potential theory method was already employed to develop 3D non-axisymmetric analyses for half infinite plane cracks subjected to external loadings applied at an arbitrary point on the crack surfaces in elasticity [33], piezoelectricity [34] and piezo-thermo-elasticity [35]. However, there is no report yet in academic records, on the non-axisymmetric analyses within the framework of thermoelasticity for half infinite plane crack.

This paper aims to make exact and complete 3D analyses for transversely isotropic media containing a half infinite plane crack subjected to temperature loads symmetrically applied on the

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upper and lower crack surfaces. The original problem is turned into a mixed boundary value problem in view of the symmetry inherent to the problem. The transformed problem is solved by the generalized potential theory method conjugated with the general solutions. A new potential function is introduced to consider the thermal effect. The integral and integro-differential equations involved in the present study are, respectively similar to the governing equations for punch and crack problems in pure elasticity. The governing equations are solved by directly employing the results available in the literature. For a point temperature load, the corresponding Green's functions along with their derivatives of various orders are derived. Exact and complete fundamental solutions are constructed in terms of elementary functions, *for the first time*. The singular behavior of the crack tip is examined and the stress intensity factor is given explicitly. For an arbitrary distributed thermal load, the stress intensity factor can be determined by an integral and an application of the current fundamental solutions has been presented for a particular plane strain problem. In the present study, the temperature field is derived via two different ways and a perfect agreement is achieved. In view of the merits mentioned above, the present solution can serve as a benchmark to various simplified analyses and numerical codes.

2. Steady state general solutions for thermoelasticity

In Cartesian coordinate system $Oxyz$, the constitutive equation for transversely isotropic media with the isotropic plane perpendicular to the z-axis is described by the Duhamel–Neumann relations [12,24]

$$\sigma_x = c_{11}\frac{\partial u}{\partial x}+c_{12}\frac{\partial v}{\partial y}+c_{13}\frac{\partial w}{\partial z}-\beta_1 T,\quad \tau_{yz}=c_{44}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),$$
$$\sigma_y = c_{12}\frac{\partial u}{\partial x}+c_{11}\frac{\partial v}{\partial y}+c_{13}\frac{\partial w}{\partial z}-\beta_1 T,\quad \tau_{zx}=c_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),$$
$$\sigma_z = c_{13}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+c_{33}\frac{\partial w}{\partial z}-\beta_3 T,\quad \tau_{xy}=c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \tag{1}$$

where β_i and c_{ij} are, respectively thermal moduli and elastic constants with the identity $2c_{66}=c_{11}-c_{12}$ holding; $u(v,w)$ and $\sigma_{ij}(\sigma_i,\tau_{ij})$ are the displacement and stress components, respectively; T is the temperature variation with $T=0$ corresponding to the stress-free state.

Substitution of the constitutive relations in Eq. (1) into the equilibrium equations ($\sigma_{ji,j}=0$) gives rise to

$$\left(\frac{c_{11}+c_{66}}{2}\Delta+c_{44}\frac{\partial^2}{\partial z^2}\right)U+\frac{c_{11}-c_{66}}{2}\Lambda^2\overline{U}+(c_{13}+c_{44})\Lambda\frac{\partial w}{\partial z}-\beta_1\Lambda T=0,$$
$$\left(c_{44}\Delta+c_{33}\frac{\partial^2}{\partial z^2}\right)w+\frac{c_{13}+c_{44}}{2}\frac{\partial}{\partial z}(\overline{\Lambda}U+\Lambda\overline{U})-\beta_3\frac{\partial T}{\partial z}=0, \tag{2}$$

where $\Lambda=\partial/\partial x+i\partial/\partial y$, $\Delta=\Lambda\overline{\Lambda}=\partial^2/\partial x^2+\partial^2/\partial y^2$, $U=u+iv(i=\sqrt{-1})$ and the over bar represents the complex conjugate of the indicated quantity.

The temperature field for the medium in a steady-state is governed by

$$\left(k_{11}\Delta+k_{33}\frac{\partial^2}{\partial z^2}\right)T=0 \tag{3}$$

where k_{ij} is the thermal conductivity coefficient.

The general solutions to Eqs. (2) and (3) proposed by Chen et al. [24] are

$$u=\frac{\partial\psi_0}{\partial y}-\sum_{j=1}^{3}\frac{\partial\psi_j}{\partial x},\quad v=-\frac{\partial\psi_0}{\partial x}-\sum_{j=1}^{3}\frac{\partial\psi_j}{\partial y},$$

$$w=\sum_{j=1}^{3}\alpha_{i1}\frac{\partial\psi_j}{\partial z_j},\quad T=\sum_{j=1}^{3}\alpha_{i2}\frac{\partial^2\psi_j}{\partial z_j^2}, \tag{4}$$

where $\psi_j(j=0,1,2,3)$ are quasi-harmonic functions

$$\left(\Delta+\frac{\partial^2}{\partial z_j^2}\right)\psi_j=0,\quad (j=0,1,2,3); \tag{5}$$

$z_j=zs_j$ and $s_j(j=0,1,2,3)$ are material eigenvalues; $s_0=\sqrt{c_{66}/c_{44}}$, $s_3=\sqrt{k_{11}/k_{33}}$, s_1 and s_2 are the roots with a positive real part of the following algebraic equation

$$a_0s^4-b_0s^2+c_0=0. \tag{6}$$

It is noted that Eq. (4) is valid only in the case of distinct material eigenvalues.

Introducing the following complex variables

$$\sigma_1=\sigma_x+\sigma_y,\quad \sigma_2=\sigma_x-\sigma_y+2i\tau_{xy},\quad \tau_z=\tau_{xz}+i\tau_{yz}, \tag{7}$$

one can derive the thermoelastic field in the compact form as follows

$$U=-\Lambda\left(\sum_{j=1}^{3}\psi_j+i\psi_0\right),\sigma_z=\sum_{j=1}^{3}\gamma_{j1}\frac{\partial^2\psi_j}{\partial z_j^2},$$
$$\sigma_1=\sum_{j=1}^{3}\gamma_{j2}\frac{\partial^2\psi_j}{\partial z_j^2},\sigma_2=-2c_{66}\Lambda^2\left(\sum_{j=1}^{3}\psi_j+i\psi_0\right),$$
$$\tau_z=\Lambda\left(\sum_{i=j}^{3}\gamma_{j3}\frac{\partial\psi_j}{\partial z_j}-is_0c_{44}\frac{\partial\psi_0}{\partial z_0}\right). \tag{8}$$

The constants α_{ij}, $a_0(b_0,c_0)$ and γ_{ij} involved in Eqs. (4), (6) and (8) are given in Appendix A.

Early researches [22] clearly revealed that the general solutions along with the potential theory method will definitely facilitate solving the mixed boundary value problems arising in the crack and contact analyses, especially the non-axisymmetric problems. Some non-classic 3D fundamental solutions were thus developed. To show the versatility of the potential theory method, an infinite body weakened by a half-infinite plane crack is considered in the next section.

3. Generalized potential method for half-infinite plane crack

Consider an infinite transversely isotropic thermoelastic body containing a half-infinite plane crack whose surface is parallel to the plane of isotropy (see Fig. 1). The coordinate system is established such that the xoy plane is coincident with the crack surface, and the origin O locates at the edge of the crack. Two symbols S and $\overline{S}$ are introduced to denote the regions on the plane $z=0$(denoted by I), respectively occupied by the crack and its complement and $S\equiv\{(x,y)|0\le y<\infty,\ -\infty<x<\infty\}$, $S\cup\overline{S}=I$ and $S\cap\overline{S}=\emptyset$, implying no intersection and separation. A pair of

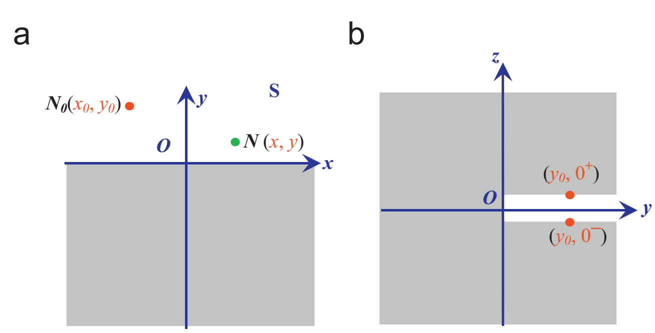

Fig. 1. Horizontal (a) and vertical (b) cross sections of a half infinite crack.