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Inelastic response of solids described by implicit constitutive relations with nonlinear small strain elastic response

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A R T I C L E I N F O

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A B S T R A C T

We develop a model to describe the inelastic response of bodies that exhibit non-linear response even in the small strain regime. We introduce a new approach to modeling the inelastic response of materials by gainfully exploiting the discontinuity of the functions that appear in the constitutive relations to describe a plethora of inelastic responses that have been observed. The model that has been developed and its generalizations can be used to describe the response of geomaterials such as clay and rocks, traditional materials such as Aluminum and several polymeric solids, as well as modern intermetallic alloys.

1. Introduction

The aim of this work is to introduce, in the simplest possible terms, a novel model for the rate independent[1] inelastic response of materials that may exhibit (1) nonlinear elastic response even for small deformations and (2) display strongly hysteretic and rate independent response with or without a yield point. Traditionally, such materials have included metals like Aluminum as well as certain soils or clays. Recently, a new class of intermetallic alloy of Titanium, Tantalum, Niobium, Vanadium and Zirconium, called "gum metal" which has several unique characteristics such as elastic response up to 2% strain, and almost perfectly plastic response with no sharp yield point, has been synthesized (Saito et al., 2003). The approach presented here is able to describe such materials as well as a wide range of material behavior, overcoming some difficulties that occur with conventional models wherein the stress is expressed in terms of the strain, plastic strain and other internal variables. These advantages include.

1. Not needing to introduce the notion of plastic strain.[2] This is important for materials that do not show any discernable elastic range (such as metals like Aluminum, many polymers and soils) but still show rate independent hysteretic response.

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[1] By rate independent we mean a response such that the path in the state space is unaltered by changing the time scale by any positive constant factor. Operationally, for the models considered here it means that the response functions are positively homogenous of order one in the rates of the state variables.

[2] We will use commonly accepted terminology such as "plastic strain", even though a more precise description would be "the strain that is not recovered upon the instantaneous removal of loads" As can be seen here, the very fact that other meanings can be attributed to "plastic strain" such as "flow of material through a crystal lattice in crystal plasticity") indicates that it would be useful to develop a theory where this notion is not a fundamental ingredient but a result of a solution of a boundary value problem.

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2. Capability to build a consistent nonlinear model within the context of small deformations where the strains are linearized strains but the relation between the stress and the linearized strain is non-linear. This is not consistently possible where an explicit expression for the stress in terms of the strain is introduced; however, a systematic process can be carried out (see Rajagopal, 2003; Rajagopal and Srinivasa, 2009; Rajagopal, 2011) when the stress is written in an implicit form.
3. Possibility of developing a model with *no elastic range* so that the response is always hysteretic and rate independent even for small deformations.
4. Ease of solution of the system of ordinary differential equations.

In order to illustrate the major points and to keep the discussion short, we will consider only a one-dimensional purely mechanical model. The additional considerations needed to develop a three dimensional theory are beyond the scope of this short paper.

2. Need for response relations that can describe the rate independent inelastic response of materials with and without yield surfaces

2.1. General considerations

The theoretical foundations of conventional inelasticity and damage mechanics has been based on the following paradigms:

- A notion of “plastic strain” or “plastic deformation”. This means different things in different contexts as noted by Naghdi (1990) and is especially troublesome when one considers thermo-inelastic response.[3]
- A Green elastic constitutive relation.
- A yield condition that separates elastic and inelastic regimes of behavior.
- A loading condition that is used for deciding when the material undergoes elastic and plastic behavior
- A flow rule that describes how the plastic strain evolves.

These ideas have been the subject of substantial debates since the 1950s both from the theoretical viewpoint (especially the notion of plastic deformation, the conditions for loading, and the form of the flow rule) as well as from an experimental viewpoint (non-existence of a sharp yield point for most materials, complex loading and unloading criteria, etc.) and from a computational viewpoint (including the use of hypoelastic constitutive laws). Nevertheless, models that deviate significantly from the core paradigm have come in for severe criticism from the computational mechanics standpoint due to the supposed lack of a thermodynamical framework (see Simo and Pister (1984) as well as Simo and Hughes (1998)).

Furthermore, there appears to be no clear consensus as to what is meant by plastic strain and this notion has been used differently in different settings (see Naghdi (1990) for a description of the different ways of “defining” plastic strain. See also Green and Naghdi (1973)).

Furthermore, recent work on finite elasticity have called into question even basic tenets of hyperelasticity (Rajagopal, 2003, (2007)), including the very basic issues what constitutes an elastic material (Rajagopal, 2007; Rajagopal & Srinivasa, 2009). For example, from a thermodynamical point of view, it seems more appealing to consider a material as being elastic if it is non-dissipative (i.e. incapable of converting mechanical work into heat). As shown by Rajagopal and Srinivasa (2009), this leads to a much larger class of materials than conventional Green elasticity, subsuming the latter as a special case. A noteworthy feature of this generalization is that *these models do not require that the stresses be expressed explicitly as functions of the deformation gradient or a suitable measure of the strain, or vice versa*. Instead, just as is done in classical thermodynamics, we obtain implicit constitutive relations, i.e., some function of stress and strain is zero.

2.2. Applications to soil and rock mechanics

Specifically, in the soil and rock mechanics literature, the notion of an elastic regime and a “plastic strain” has always been viewed with some unease and many attempts have been made to eliminate such a demarcation (see e.g. Mroz (1980), Pastor et al. (1990), Lubliner (1991)). In the same vein, the notion of a yield surface that separates purely hyperelastic and plastic responses which makes eminent sense for materials like steel where there is a sharp yield surface, is quite questionable when applied to concrete and soil mechanics where the material exhibits hysteretic behavior throughout its deformation.

2.3. Applications to metal forming and related fields

Similarly in the metal forming literature, due to the exigencies of the technology, it is common to identify the plastic strain as the total strain. Indeed many commercial metal forming softwares completely ignore the notion of a plastic strain and

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[3] For example, if we assume that the plastic strain is obtained by unloading, then an unloading protocol or path has to be specified (examples of which include isothermal unloading and adiabatic or isentropic unloading). The resulting plastic strain is different depending upon the protocol.