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A unified residual-based thermodynamic framework for strain gradient theories of plasticity

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ABSTRACT

A unified thermodynamic framework for gradient plasticity theories in small deformations is provided, which is able to accommodate (almost) all existing strain gradient plasticity theories. The concept of *energy residual* (the long range power density transferred to the generic particle from the surrounding material and locally spent to sustain some extra plastic power) plays a crucial role. An *energy balance principle* for the extra plastic power leads to a *representation formula* of the energy residual in terms of a *long range stress*, typically of the third order, a macroscopic counterpart of the micro-forces acting on the GNDs (Geometrically Necessary Dislocations). The *insulation condition* (implying that no long range energy interactions are allowed between the body and the exterior environment) is used to derive the higher order boundary conditions, as well as to ascertain a *principle of the plastic power redistribution* in which the energy residual plays the role of redistributor and guarantees that the actual plastic dissipation satisfies the second thermodynamics principle. The (nonlocal) Clausius–Duhem inequality, into which the long range stress enters aside the Cauchy stress, is used to derive the thermodynamic restrictions on the constitutive equations, which include the state equations and the dissipation inequality. Consistent with the latter inequality, the evolution laws are formulated for rate-independent models. These are shown to exhibit multiple size effects, namely (energetic) size effects on the hardening rate, as well as combined (dissipative) size effects on both the yield strength (intrinsic resistance to the onset of plastic strain) and the flow strength (resistance exhibited during plastic flow). A friction analogy is proposed as an aid for a better understanding of these two kinds of strengthening effects. The relevant boundary-value rate problem is addressed, for which a solution uniqueness theorem and a minimum variational principle are provided. Comparisons with other existing gradient theories are presented. The dissipation redistribution mechanism is illustrated by means of a simple shear model.

1. Introduction

The strain gradient plasticity theory herein presented stems as an evolution of the "residual-based" theory by the author and co-workers, with notable extensions and improvements in regard to the basic task to capture and predict size scale phenomena. The residual-based theory has been shown to be able to cope with plastic strain localization problems (Polizzotto and Borino, 1998; Polizzotto, 2003b), as well as with internal and boundary interfaces (Polizzotto, 2008, 2009a); also, it has been shown to be able to capture and describe the so-called energetic size effects (Polizzotto, 2007; Borino and Polizzotto, 2007) as well as dissipative size effects on the yield strength (Polizzotto, 2010). Moreover, it has been extensively compared

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with the analogous theory based on the virtual work principle (VWP) (Gurtin, 2003, 2004; Gurtin and Anand, 2005; Anand et al., 2005; Fleck and Willis, 2009a,b; Gudmundson, 2004; Fredriksson and Gudmundson, 2007; Abu Al-Rub et al., 2007) in the papers by Borino and Polizzotto (2007), Polizzotto (2008, 2009b).

It emerged that, although there exists a strict link between the two theories, the residual-based theory proves to be less general than the VWP-based one mainly because the former admits only conventional plastic dissipation modes, i.e. of the type $D = \sigma^*_{ij}\dot{\varepsilon}_{ij}$ (Polizzotto, 2009b), so leaving out materials with higher order dissipation modes of the type $D = \tau^*_{kij}\dot{\varepsilon}_{ij,k}$, which are of particular concern within gradient plasticity (Gurtin, 2004; Gurtin and Anand, 2005; Fleck and Willis, 2009a,b).

In the present extended version of the residual-based theory, not only the aforementioned restriction on the plastic dissipation modes is removed, but also a few new physically meaningful concepts are introduced, aside the original one, i.e. the *nonlocality energy residual*, or simply *residual*. As it will be explained shortly, all these provisions make the extended residual-based theory constitute a unified framework, consistent with thermodynamics principles, able to accommodate (almost) all other existing gradient theories. Gradient crystal plasticity models may also be accommodated, but for simplicity the latter models are not considered here.

In order to explain the key ideas, on which the proposed theory is grounded, an elastic plastic body subjected to quasi-static varying loads is considered in the conventional linearized small deformation regime. The existence of internal and/or boundary interfaces – often considered in the literatures (see e.g. Aifantis et al., 2006; Borg and Fleck, 2007; Fleck and Willis, 2009b) to account for grain boundary effects – will be taken into consideration in a future paper.

The *energy residual*, say *P*, on which the proposed theory is grounded, accounts for the microstructural long range interaction processes promoted by inhomogeneities and defects with the inherent high plastic strain gradients. *P* equals the long range power density transmitted to the generic particle from all other particles in the body; it is locally spent to promote an *extra plastic power* which is accomplished by some *inner* stresses and adds to the conventional plastic power accomplished by the Cauchy stress, $\boldsymbol{\sigma}$. The energy residual constitutes a paramount feature for a nonsimple (either gradient type, or integral type) material, meaning that the material is simple if, and only if, *P* is identically vanishing for whatever deformation mechanism.

The concept of energy residual was introduced within the framework of general nonlocal continuum theories (Edelen and Laws, 1971; Eringen and Edelen, 1972). Point-wise balance equations enriched by several nonlocality residuals (as mass residual, body force and body couple residuals, energy residual) were there used with the intent to macroscopically account for the long range interatomic interactions (see e.g. Eringen, 2002, and references cited therein). As a matter of facts, the resulting nonlocal theories proved to be rather cumbersome for the many parameters needing experimental verifications. Additionally, whereas the mass residual can be considered null in the absence of chemical processes, of all nonlocality residuals only the energy residual is an objective quantity (i.e. it leaves the relevant balance equation invariant under any rigid-body transformation) and it thus is the only one to be retained for a consistent treatment of the constitutive behavior of nonsimple materials. This was done by Dunn and Serrin (1985) for Korteweg materials and by Maugin (1990) for gradient materials, but none of them enforced the *insulation condition* previously advanced by Edelen and Laws (1971). A single energy residual in conjunction with the insulation condition was subsequently used by Polizzotto and Borino (1998), who coined the term "insulation condition", and Polizzotto (2003a, 2007) for gradient plasticity, by Borino et al. (1999) for nonlocal plasticity, by Polizzotto (2001) for nonlocal elasticity, by Polizzotto (2003a) for gradient elasticity and by Liebe and Steinmann (2001), Benvenuti et al. (2002), Borino et al. (2003) for damage mechanics, and others.

The proposed theory is based upon the following constitutive assumptions:

- The *insulation condition*: The body is a *constitutively closed system*, therefore no long range interactions can occur between the material particles and the exterior environment, or, in other words, the global long range energy supplied to the entire body is vanishing, that is

$$P_{\rm tot} = 0. \tag{1}$$

- The *locality recovery condition*: The material has to behave as a simple material whenever the plastic strain field is homogeneous. A thermodynamically consistent way for this is that the energy residual *P* is vanishing identically whenever the plastic strain $\boldsymbol{\varepsilon}^{\rm p}$ is uniform in *V*, namely

$$P = 0 \text{ in } V \quad \forall \boldsymbol{\varepsilon}^{\rm p} \text{ uniform in } V. \tag{2}$$

This condition, devised by Polizzotto et al. (2006) for nonlocal plasticity, ensures that the material response is correspondingly simple both in stress and in energy.

In the proposed theory, an *energy balance principle for the extra plastic power* is established. This principle is used to derive a notable *representation formula* for the residual *P* in terms of a *long range stress*, say $\boldsymbol{\tau}$. This formula is found to incorporate entirely the basic information contained in the so-called "microstress equilibrium equations" which in other theories of gradient plasticity are derived by means of the extended VWP previously mentioned. The above balance principle is also used to readily derive this VWP, showing that the long range stress is the same thing as the higher order microstress of the VWP-based theory.

It is also found that the long range stress $\boldsymbol{\tau}$ is typically of the third order, but even a vector when the plastic deformation mechanism is described by a simple scalar variable (effective plastic strain). Moreover, the long range stress is in general