International Journal of Pressure Vessels and Piping 120-121 (2014) 27–35

Contents lists available at ScienceDirect

International Journal of Pressure Vessels and Piping

journal homepage: www.elsevier.com/locate/ijpvp

Stresses in heated pressurized multi-layer cylinders in generalized plane strain conditions

Knut Vedeld*, Håvar A. Sollund

Mechanics Division, Department of Mathematics, University of Oslo, Moltke Moes vei 35, PO Box 1053 Blindern, 0316 Oslo, Norway

ARTICLE INFO

Article history:
Received 10 January 2014
Received in revised form
24 April 2014
Accepted 28 April 2014
Available online 9 May 2014

Keywords:
Multi-layer cylinders
Generalized plane strain
Thermal stresses
Pressure vessels
Pipes

ABSTRACT

An analytical solution for the displacement field and corresponding stress state in multi-layer cylinders subjected to pressure and thermal loading is developed. Solutions are developed for axially loaded and spring-mounted cylinders, assuming that the combined multi-layer cross-section remains plane after deformation (generalized plane strain). The analytical solutions are verified by means of detailed three-dimensional finite element analyses. The solutions are easily implemented in, and suitable for, engineering applications.

1. Introduction

Cylinders subjected to pressure and temperature have been widely studied, and an analytical solution for the displacement field of a linearly elastic, isotropic hollow cylinder exposed to uniform internal and external pressure was derived by the French mathematician Gabriel Lamé already in 1831 [1]. The solution suggested by Lamé is readily adapted for application to shrink-fit problems [2]. Since Lamé's solution gives the full three-dimensional stress state of a pressurized cylinder, the solution is highly useful for design of pressurized thick-walled cylindrical members or disks. However, conventional single-layer pressure vessels are often not suited for operation in extreme environments, with conditions characterized by either high pressures and high temperatures, or potentially strongly corrosive fluid contents or combinations thereof [3–6]. Other reasons to include more layers may be thermal insulation and external corrosion protection [7–9]. By increasing the number of cylindrical layers, a larger number of design variables become available to the designer. Naturally, this flexibility facilitates more optimal design for each specific application. Two-layer and multi-layer cylinder designs are therefore often utilized for e.g., pressure vessels [10,11], pipelines [5,12], piping systems [13,14] and risers [15]. On the other hand, the introduction of additional layers results in increased computational complexity [16] and new potential failure mechanisms to control [3,9,17].

Mechanical response and thermoelastic properties of layered cylinders have been studied extensively due to their frequent application in industrial design. Due to corrosion resistant liners or cladding, weight coatings, external corrosion coatings and insulation coatings, piping systems and offshore pipelines are always layered, and design of pipelines and piping systems rely heavily on the mechanical and thermoelastic response of cylinders, as evident from governing design codes such DNV-OS-F101 [18], API RP 1111 [19] and ASME B31.8 [20]. Auto-frettage and shrink-fit techniques are highly common for production of layered cylinders, resulting in research efforts toward optimization of auto-frettage design [11,21–23]. Development of more advanced manufacturing techniques has also resulted in extensive research on the mechanical and thermoelastic response of cylinders made of functionally graded materials [24–27]. Fatigue and capacity assessment of layered cylinders subjected to thermal shock and series of micro shocks from time-dependent flow temperature and density characteristics, constitute a challenge for piping systems, particularly with multi-phase flow, as detailed by Radu et al. [28] and Marie [13]. Thermal loading has been treated for a variety of conditions in multi-layered cylinders. Uniform thermal stresses were applied by Akcay and Kaynak [29], and loading from steady-state temperature distributions has been studied extensively [6,24,30]. Time-dependent thermal stresses, both transient [28,31–33] and cyclic [34], have also been widely covered. Other multi-layer systems,

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* Corresponding author.
E-mail addresses: knutved@math.uio.no, knut.vedeld@dnv.com (K. Vedeld).
URL: http://www.math.uio.no

http://dx.doi.org/10.1016/j.ijpvp.2014.04.002
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Nomenclature

a_i	recurrence relation, defined by Eq. (27) [−].
A_i	$= -\widehat{E}_i(1-2\nu_i)C_{r1,i}$[N]
$A_{s,i}$	steel cross-sectional area for layer i [m^2]
b_i	recurrence relation, defined by Eq. (27) [−].
C	general constant (used for strain) [−]
C $[a, b]$	space of real continuous functions on the interval $[a, b]$
c_i	recurrence relation, defined by Eq. (38) [−].
C_i	$= \widehat{E}_i C_{r2,i}$ [Pa]
$C_{r1,i}$	displacement coefficient in radial direction for layer i [m^2]
$C_{r2,i}$	displacement coefficient in radial direction for layer i [−]
C_z	displacement coefficient in axial direction [m]
d_i	recurrence relation, defined by Eq. (38) [−]
E_i	Young's modulus for layer i [Pa]
$\widehat{E}_i$	$=E_i/((1+\nu_i)(1-2\nu_i))$ [Pa]
i	layer index [−]
K	axial spring stiffness [N/m]
L	length of cylinder
n	total number of layers [−]
N	applied axial load [N]
p_{ext}	external pressure [Pa]
p_{int}	internal pressure [Pa]
q_0	$=p_{\text{int}}$ [Pa]
q_i	contact pressure between layer i and $(i+1)$ [Pa]
$q^0{}_i$	contact pressure between layer i and $(i+1)$ for plane strain [Pa]
q_n	$=p_{\text{ext}}$ [Pa]
r	radial coordinate [−]
r_0	inner radius of combined cross-section [m]
r_i	outer radius of cylinder layer i [m]
r_n	outer radius of combined cross-section [m]
S_i	$=(\lambda_i-\beta_{i+1})\gamma_{i+1}\gamma_i+(\lambda_{i+1}-\lambda_i)\gamma_i$ [Pa^{-1}]
T_i	$=\lambda_{i+1}-\beta_i+(\beta_i-\beta_{i+1})\gamma_{i+1}$ [Pa^{-1}]
$u_{r,i}$	displacement field component in radial direction for layer i [m]
u_z	displacement field component in axial direction [m]
$u_{z,i}$	displacement field component in axial direction for layer i [m]
$u_{\theta,i}$	displacement field component in circumferential direction for layer i [m]
x	Cartesian coordinate [m]
y	Cartesian coordinate [m]
z	Cartesian/cylindrical coordinate [m]
α_i	temperature expansion coefficient for layer i [°C^{-1}]
β_i	$= 1/\widehat{E}_i$ [Pa^{-1}]
γ_{i+1}	$= r_i^2/r_{i+1}^2$ [−]
$\gamma_{ij,i}$	shear strains in layer i [−]
ΔT_i	change in temperature in layer i [°C]
$\varepsilon_{ij,i}$, $\boldsymbol{\varepsilon}_i$	strain tensor for layer i [−]
ε_{zz}	$=C_z/L$ (axial strain) [−]
ζ_i	$=(d_i-(d_n/a_n)a_i)q_0$ [Pa]
θ	circumferential coordinate [−]
λ_i	$= -1/\widehat{E}_i(1-2\nu_i)$ [Pa^{-1}]
μ_i	$= \phi_i - \widehat{E}_i\nu_i\varepsilon_{zz}$ [Pa]
ν_i	Poisson's ratio for layer i [−]
σ_{ij}, $\boldsymbol{\sigma}$	stress tensor [Pa]
$\sigma_{ij,i}$, $\boldsymbol{\sigma}_i$	stress tensor for layer i [Pa]
φ_i	$= \widehat{E}_i\alpha_i\Delta T_i(1+\nu_i)$ [Pa]

including films, ceramics and coatings in microelectronic, optical and structural components have been studied as well [16]. With regard to axial restraints, the studies on multi-layered or thick-walled cylinders have generally been restricted to either plane stress (no friction between the layers) [23,31,33,35] or plane strain conditions (no axial strain) [26,29,36], or both plane stress and plane strain [27,37].

For realistic conditions, layered cylinders such as pipelines, piping systems, risers and tanks may be in plane strain and will generally not be in plane stress conditions [9,17,38,39]. Some degree of (or most often complete) axial fixation between layers restricts independent axial displacement of individual layers making the plane stress condition unrealistic. In some cases, cylinders are fixed axially, which results in plane strain. Most often, however, some or full axial freedom of the combined cross-section applies, resulting in various conditions of generalized plane strain.

In the present study, the elastic response of multi-layer hollow cylinders in generalized plane strain conditions subjected to pressure and temperature will be investigated. Physically transparent equilibrium and kinematic compatibility requirements will be employed to establish a second-order difference equation in terms of the contact stresses between the individual layers. By solving for the contact stresses, a recursive algorithm may be developed for the calculation of stresses and displacements in each cylinder layer. Such a technique was first adopted by Xiang et al. [27] and Shi et al. [37], who studied pressurized multi-layer cylinders, but considered neither temperature effects nor generalized plane strain conditions. In the present manuscript, it will be demonstrated how the inclusion of thermal stresses transforms the second-order difference equation for the contact stresses from a homogenous equation into an inhomogeneous one, and how the solution of the equation for the case of generalized plane strain gives rise to a formula for the axial strain of the cross-section. The results of the novel recursive algorithm will be compared to results from finite element analyses (FEA).

The novel recursive algorithm is highly useful since it is computationally efficient, easily implemented in spreadsheet-based calculations and requires no advanced calculation programs to be applied. Due to its efficiency, any number of layers may be implemented with negligible computational effort. Hence, variation in material characteristics or temperature distributions can be considered to any desired accuracy.

2. Problem definition

2.1. A priori assumptions

(i) The materials in the cylinder layers are assumed to be linearly elastic, homogenous and isotropic.
(ii) Initial stresses and strains are disregarded.
(iii) Small displacements are assumed. Thus, the load is applied on the initial geometry, and changes in internal or external diameter and changes in layer wall thickness due to the application of loading are not accounted for.
(iv) Heat is assumed to result in a uniform temperature distribution within each cylinder (sub-)layer.