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Application of conservation laws to Dirichlet problem for elliptic quasilinear systems

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ABSTRACT

In the present work we show the possibility of using of conservation laws to solve the Dirichlet problem for elliptic quasilinear systems. As a result the integral representation of solution is obtained. For the system of filtration of aerated fluid in porous medium and for system of elastic–plastic torsion of prismatic rods corresponding conservation laws are calculated in explicit form and the Dirichlet problems are solved.

1. Introduction

The present article is in a sense a continuation of our works on the application of conservation laws (CL) to solve boundary value problems (BVP) of quasilinear systems of partial differential equations (PDEs). In [15,16] it was shown that the method of CL permits to avoid the restriction on the Jacobian of hodograph transformation, usually using to solve hyperbolic quasilinear PDEs. The existence of two families of real characteristics has played a significant part and the analogue of Riemann–Green function for quasilinear system has been constructed. This method of CL was used for solution of a variety of boundary problems and has recently obtained some extension in a series of papers [18–20] (see also [9]), where authors proposed a method which allows to reduce the Cauchy problem for the two quasilinear PDEs to the Cauchy problem for ordinary differential equations.

Here we show the possibility of applying of CL to solve a BVP for quasilinear elliptic PDEs. The special attention is made for the problem with unknown boundary in particular for elastic–plastic torsion of prismatic rods.

The concept of conservation law is of the fundamental importance in the theory of differential equations. The formal definition of this concept from symmetry groups point of view can be found in books [2,3,10,11] and references within. There appear two principal directions of investigations. The first one is the mathematical description of conservation laws of a given system (for instance [13,14]) and the second one is using CL to determine solutions of PDEs in practical applications. There is a lot of results in both directions. For recent advances see for example the collection of articles in [4].

Application of CL to solve boundary problems is still an actual problem. This problem was posed in 1983 by Vinogradov and Senashov and is realized in the series of works mentioned above. Moreover, Shemarulin [17] proposed an effective method for solving boundary-value problems for linear PDEs using a complete system of conservation laws. In such a way the problem for the system of fluid filtration in cleaved-porous medium was solved.

Let us consider a quasilinear autonomous system of PDEs of two independent variables x, y and two dependent ones u, v in form [12]:

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = F, \tag{1}$$

where $A = \|a_{ij}(u, v)\|$, $B = \|b_{ij}(u, v)\|$, $F = (f_1, f_2)^T$, $f_i = f_i(u, v)$, $i, j = 1, 2$, $U = (u, v)^T$.

If matrix A is not degenerate, then system (1) can be written in the normal form

$$\frac{\partial U}{\partial x} + M\frac{\partial U}{\partial y} = C, \tag{2}$$

where $M = A^{-1}B = \|m_{ij}(u, v)\|$ is a real-valued matrix, $C = A^{-1}F = (c_1, c_2)^T$.

Let system (2) be an elliptic one. It means that matrix M has

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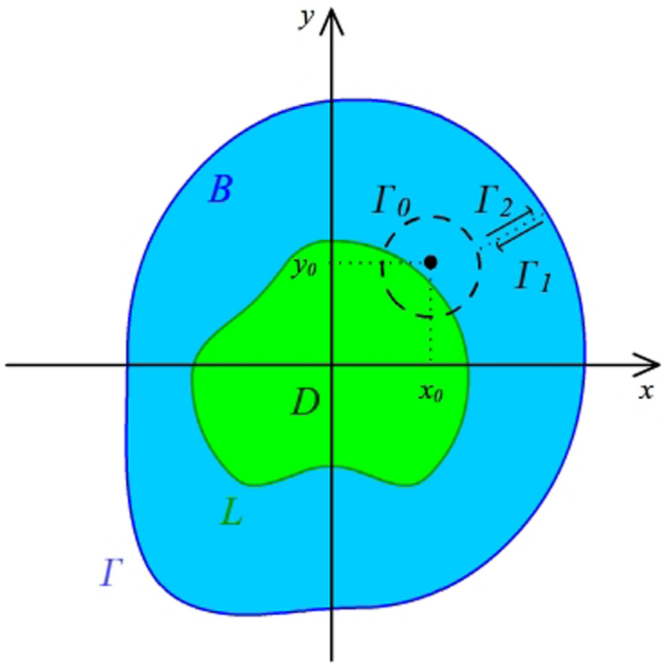

Fig. 1. Dirichlet problem for quasilinear elliptic system.

two complex conjugated eigenvalues $\lambda_{1,2}(u, v)$ obtained as roots of the equation:

$$\det(M - \lambda E) = 0: \quad 2\lambda_{1,2} = m_{11} + m_{22} \pm i\sqrt{|\Delta|},$$

where $\Delta = (m_{11} - m_{22})^2 + 4m_{12}m_{21} < 0$ and necessary $m_{12}m_{21} < 0$.

Let us mention the recent article [5], where the so-called symmetry reduction method (using CL) is applied to solve first-order quasilinear elliptic systems.

Let us set up the inner Dirichlet problem due to [1]. Let $\Gamma = \{x = f(t),\ y = g(t),\ t \in (t_1, t_2)\}$ be the piecewise smooth simple closed boundary of simply-connected region $S = B\bigcup D$ (see Fig. 1). It is necessary to determine the solution of (2) at point $(x_0, y_0) \in S$ satisfying the boundary condition along Γ

$$U(x, y)|_\Gamma = U_\Gamma(x, y) = (u_\Gamma, v_\Gamma). \tag{3}$$

2. Conservation laws

Conservation laws of system (2) is a divergence expression

$$D_x\varphi + D_y\psi = 0,$$

which is valid for any solution of (2) and D_x, D_y are the operators of total derivative on x and y:

$$D_x = \frac{\partial}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} + \cdots, \quad D_y = \frac{\partial}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} + \cdots$$

There are different manners of constructing CL (see the bibliography mentioned above). Each of them depends on the specificity of PDE system. Let us seek the conservation law of system (2) directly in form

$$D_x\varphi(x, y, u, v) + D_y\psi(x, y, u, v) = 0.$$

We have (hereafter variable subindex means corresponding derivative):

$$\varphi_x + \psi_y + (\varphi_u, \varphi_v)U_x + (\psi_u, \psi_v)U_y = \varphi_x + \psi_y + (\varphi_u, \varphi_v) C + \left[(\psi_u, \psi_v) - (\varphi_u, \varphi_v)M\right]U_y = 0,$$

so we obtain the linear system of three equations

$$\varphi_x x + \psi_y = -(\varphi_u, \varphi_v)C, \quad (\psi_u, \psi_v) = (\varphi_u, \varphi_v)M. \tag{4}$$

Let us take solution

$$\varphi = \varphi_1(x - x_0, y - y_0, u, v), \quad \psi = \psi_1(x - x_0, y - y_0, u, v)$$

of (4) that has a regular singularity at point $(x_0, y_0) \in S$. For the contour $\tilde\Gamma = \Gamma_+\bigcup\Gamma_0\bigcup\Gamma_-\bigcup\Gamma$, where $\Gamma_0 = \{(x, y): (x - x_0)^2 + (y - y_0)^2 = \varepsilon^2\}$ is the circumference of ε radius bounded region $\tilde S$ is simply-connected and one can apply Green theorem:

$$\iint_{\tilde S}\left(D_x\varphi_1 + D_y\psi_1\right)dxdy = \oint_{\tilde\Gamma}\varphi_1 dy - \psi_1 dx = 0.$$

Taking into account that $\int_{\Gamma_+}\varphi_1 dy - \psi_1 dx = -\int_{\Gamma_-}\varphi_1 dy - \psi_1 dx$ we obtain

$$\oint_{\tilde\Gamma}\varphi_1 dy - \psi_1 dx = \oint_{\Gamma_+} + \int_{\Gamma_0} + \int_{\Gamma_-} + \oint_{\Gamma} = \oint_{\Gamma_0} + \oint_{\Gamma} = 0. \tag{5}$$

Let us note that the path of integration along Γ is counter-clockwise and along Γ_0 it is clockwise.

Using parametrization of Γ_0: $x - x_0 = \varepsilon\cos t,\ \ y - y_0 = \varepsilon\sin t$, $t \in [0, 2\pi)$ we have

$$\oint_{\Gamma_0}\varphi_1 dy - \psi_1 dx = -\int_0^{2\pi}\Big[\varepsilon\varphi_1(\varepsilon\cos t, \varepsilon\sin t, u, v)\cos t + \varepsilon\psi_1(\varepsilon\cos t, \varepsilon\sin t, u, v)\sin t\Big]dt \to G_1(u_0, v_0), \quad \text{as } \varepsilon \to 0,$$

where $(u_0, v_0) = (u(x_0, y_0), v(x_0, y_0))$. Then from (5) and (3) we obtain

$$G_1(u_0, v_0) = \int_{t_1}^{t_2}\Big[\psi_1(f, g, u_\Gamma, v_\Gamma)f' - \varphi_1(f, g, u_\Gamma, v_\Gamma)g'\Big]dt. \tag{6}$$

Taking linearly independent solution (φ_2, ψ_2) of (4) with the same property of singularity at point (x_0, y_0) and proceeding in the same manner

$$\oint_{\Gamma_0}\varphi_2 dy - \psi_2 dx = -\int_0^{2\pi}\Big[\varepsilon\varphi_2(\varepsilon\cos t, \varepsilon\sin t, u, v)\cos t + \varepsilon\psi_2(\varepsilon\cos\ t, \varepsilon\sin t, u, v)\sin t\Big]dt \to G_2(u_0, v_0), \quad \text{as } \varepsilon \to 0,$$

we have

$$G_2(u_0, v_0) = \int_{t_1}^{t_2}\Big[\psi_2(f, g, u_\Gamma, v_\Gamma)f' - \varphi_2(f, g, u_\Gamma, v_\Gamma)g'\Big]dt. \tag{7}$$

Eqs. (6) and (7) implicitly define solution (u_0, v_0) of Dirichlet problem.

3. Examples of application

In this section we consider some examples of applications of the method described above.

3.1. Aerated fluid in porous medium

The system describing the steady filtration process of aerated fluid in porous medium under some assumptions can be written in the following form [7]:

$$u_x + a(u, v)v_y = 0, \quad u_y - a(u, v)v_x = 0, \tag{8}$$

where $a(u, v)$ is a smooth function determined by the properties of