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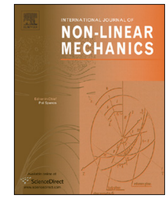

Analytical solution in parametric form for the two-dimensional free jet of a power-law fluid

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ABSTRACT

The two-dimensional free jet of an incompressible non-Newtonian power-law fluid is investigated. The Reynolds number is defined in terms of the characteristic effective viscosity of the power-law fluid. The boundary layer equations for a power-law fluid are derived in terms of the stream function. The free jet is modelled by making the boundary layer approximation perpendicular to the axis of symmetry. The conservation laws for the partial differential equation for the stream function are investigated using the multiplier method and the conserved quantity for the free jet is obtained by integrating the elementary conservation law across the jet. The Lie point symmetry of the partial differential equation for the stream function which is associated with the elementary conserved vector is derived and it is used to obtain the invariant form of the stream function. An analytical solution for the free jet in parametric form is derived. The solution depends on the exponent n in the power law. For a shear thickening fluid $(n > 1)$ it is found that the jet is bounded in the lateral direction perpendicular to the axis of the jet and the equation of the boundary is derived. For a Newtonian fluid $(n = 1)$ and a shear thinning fluid $(0 < n < 1)$ the jet is unbounded in the direction perpendicular to the axis. The solution for $n = 1/2$ is a special case and has exponential form. For $1/2 < n < \infty$ the total flux of mass along the jet and the inflow velocity at the boundary $(1 < n < \infty)$ and at infinity $(1/2 < n \le 1)$ are finite. For $0 < n \le 1/2$ the solution is not physically acceptable because the total flux of mass along the jet and the inflow velocity at infinity are infinite.

1. Introduction

In this paper we investigate the two-dimensional free jet of a power-law fluid. Industry and engineering are heavily reliant on non-Newtonian fluids, in particular, on the non-Newtonian power-law model due to its relative simplicity. Specific examples would include the extraction of crude oil from petroleum products, manufacturing of polymeric materials and inkjet printing. Power-law fluids can be described as generalised Newtonian fluids as a consequence of the variable viscosity of the fluid and the nonlinear relationship between the shear stress and strain rate. Schowalter [1] was the first to provide a derivation of the governing equations of a pseudoplastic fluid. Numerical solutions for the boundary layer equations of a power-law fluid for both shear thinning and shear thickening fluids were developed by Acrivos et al. [2]. The similarity solution of the boundary layer equations for a power-law fluid was derived by Denier and Dabrowski [3]. Two-dimensional and axisymmetric wakes of power-law fluids have been studied by Rotem [4] and Weidman and Van Atta [5]. The validity of the mathematical model for shear thinning fluids was investigated by Wu and Thompson [6].

Boundary layer theory can be extended to jet flows, which was first done by Schlichting [7,8]. He determined the numerical solution to the ordinary differential equation governing the steady flow of a two-dimensional free jet. Bickely [9] provided an analytic solution to this problem. The free jet of a non-Newtonian power-law fluid was first considered by Shul'man and Berkovskii [10]. These authors derived the similarity form of the solution and reduced the partial differential equation to an ordinary differential equation. Further, they solved the resulting differential equation for values $n = 1/2$ and $n \neq 1/2$. However, they did not consider the values $n > 1/2$ and $n < 1/2$ separately. They also did not discuss the physical interpretation of their results. Kutepov et al. [11] considered the same problem, where once again there was no distinction made for values $n > 1/2$ and $n < 1/2$ and the interpretation of the results for $n > 1$ did not predict the boundary for the jet. Lemieux and the Unny [12] considered the numerical solution of a two-dimensional free jet of a power-law fluid and were the first to confirm that a boundary curve for a two-dimensional

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free jet of a power-law fluid exists. However, these authors found that this boundary exists for all values of n. Mitwally [13] who also solved the ordinary differential equation numerically found that the boundary curve exists only for values $n > 1$ and not over the entire range $0 < n < \infty$ of n.

In this paper we derive the analytical solution in parametric form for a two-dimensional free jet of an incompressible power-law fluid. We investigate the solution and show that the jet is bounded for $n > 1$ and unbounded for $n \le 1$. We find that the similarity solution has a different form when the parameter $n < 1/2$ from that when $n > 1/2$ unlike in [10,11]. We reconfirm the result of Shul'man and Berkovskii [10], who considered the flux along the jet, that a shear thinning jet for $0 < n \le 1/2$ is not physically acceptable.

Fundamental to solving jet flow problems in fluid mechanics is the derivation of a conserved quantity. Due to the homogeneity of the boundary conditions in jet flow problems, the arbitrary parameter in the similarity solution cannot be obtained from the boundary conditions. A conserved quantity for the jet is used to determine this parameter and also to complete the solution by obtaining the remaining constant of integration. For boundary layer flow past a solid boundary the mainstream matching boundary condition is not homogeneous and the arbitrary parameter can be derived. Schlichting [7] derived the conserved quantity for the two-dimensional free jet by integrating Prandtl's momentum boundary layer equation across the jet.

A significant, although challenging, requirement in the derivation of a conserved quantity is that one needs to be well acquainted with the physical properties of the problem. The difficulties are removed by using a conservation law for the partial differential equation which is a more systematic method to obtain the conserved quantity. The method which we will use to derive conservation laws is the multiplier method [14]. There are various other approaches which can be applied to obtain conservation laws [15]. The multiplier approach allows us to deduce a basis of conserved vectors and these conserved vectors together with the boundary conditions are critical in determining the conserved quantities. A significant feature of this work is that we incorporate the conservation law into the solution to the problem by reducing the partial differential equation to an ordinary differential equation with the aid of the Lie point symmetry associated with the conserved vector that generates the conserved quantity. Much attention has been given to obtaining the conserved quantities for Newtonian fluids in jet flow models. Naz et al. [14] determined the conserved quantities for both axisymmetric and two-dimensional laminar jets. For power-law fluids a systematic investigation of conservation laws and the corresponding conserved quantities for jet flow problems does not appear to have been undertaken.

Although the problem of boundary layer flow past a solid boundary has usually been solved numerically, jet flow problems for Newtonian fluids can generally be solved analytically. This includes free jets, liquid jets and wall jets [16]. Recently Lie group analysis has been applied to solve jet flow problems for Newtonian fluids in which a linear combination of the Lie point symmetries of the partial differential equation is used to derive a group invariant solution. Mason [17] and Mason and Hill [18] used this approach to derive group invariant solutions for laminar and turbulent two-dimensional free jets. A more efficient method to obtain the invariant solution is to first derive the Lie point symmetry associated with the conserved vector which yields the conserved quantity for the jet and to use this Lie point symmetry to generate the invariant solution. The relationship between Lie point symmetries and conservation laws for a partial differential equation was first derived by Kara and Mahomed [19,20]. From the double reduction theorem of Sjoberg [21,22] the ordinary differential equation obtained from the reduction of the partial differential equation can be integrated once. This approach was used by Mason and Hill [23] to derive the invariant solution for a turbulent axisymmetric free jet and by Mason and Anthonyrajah [24] to analyse turbulent flow in a pipe. We note that the solutions obtained using the Lie group method are not the only admissible solutions to boundary layer problems. Burde [25] found explicit similarity solutions to the boundary layer equations which cannot be obtained using the Lie group method. The approach used by Burde [25] allows for the partial differential equation to be reduced to a system of ordinary differential equations rather than to a single ordinary differential equation. More general similarity transformations allow for other exact solutions which cannot be found using the standard Lie group method [26].

The ordinary differential equations obtained by reducing the partial differential equations for the two-dimensional free jet of a Newtonian fluid and power-law fluid are autonomous and as such can be solved analytically. The axisymmetric free jet for a Newtonian fluid and a power-law fluid are described by non-autonomous ordinary differential equations. For the Newtonian fluid the non-autonomous ordinary differential equation can be solved analytically [8] but for the power-law fluid no analytical solution has been derived and it is solved numerically. Serth [27] derived the numerical solution for an axisymmetric free jet of a power-law fluid. Mitwally [13] obtained numerical solutions to the two-dimensional and axisymmetric free and wall jets of a power-law fluid.

An outline of the study is as follows. In Section 2, we present a thorough derivation of the mathematical model for the two-dimensional free jet of a non-Newtonian power-law fluid. The conservation laws and the conserved quantities for the two-dimensional free jet are obtained in Section 3. We derive the associated Lie point symmetry and deduce the form of the invariant solution in Section 4. In Section 5 analytical solutions in parametric form are presented for the whole range of shear thinning and shear thickening power-law fluids. The properties of the analytical solutions are analysed in Section 6. Finally the conclusions are drawn in Section 7.

2. Mathematical model

Consider a steady two-dimensional free jet consisting of an incompressible power-law fluid. The jet emerges from a long narrow orifice in a wall into surrounding fluid which consists of the same power-law fluid at rest. The x-axis is along the axis of symmetry of the jet and the origin of the coordinate system is at the orifice. We present a complete derivation of the model in order to make the paper self-contained and to clarify the assumptions made in boundary layer theory for a power-law fluid. Further, we define the Reynolds number in terms of the characteristic effective viscosity.

The constitutive equation for an incompressible power-law fluid is

$$\tau_{ik} = -p\delta_{ik} + S_{ik} \tag{2.1}$$

where τ_{ik} is the Cauchy stress tensor,

$$S_{ik} = \mu_e A_{ik}, \tag{2.2}$$

A_{ik} is the rate of strain tensor,

$$A_{ik} = \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}, \tag{2.3}$$

μ_e is the effective viscosity,