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A nonlinear planar beam formulation with stretch and shear deformations under end forces and moments

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ABSTRACT

A new nonlinear planar beam formulation with stretch and shear deformations is developed in this work to study equilibria of a beam under arbitrary end forces and moments. The slope angle and stretch strain of the centroid line, and shear strain of cross-sections, are chosen as dependent variables in this formulation, and end forces and moments can be either prescribed or resultant forces and moments due to constraints. Static equations of equilibria are derived from the principle of virtual work, which consist of one second-order ordinary differential equation and two algebraic equations. These equations are discretized using the finite difference method, and equilibria of the beam can be accurately calculated. For practical, geometrically nonlinear beam problems, stretch and shear strains are usually small, and a good approximate solution of the equations can be derived from the solution of the corresponding Euler–Bernoulli beam problem. The bending deformation of the beam is the only important one in a slender beam, and stretch and shear strains can be derived from it, which give a theoretical validation of the accuracy and applicability of the nonlinear Euler–Bernoulli beam formulation. Relations between end forces and moments and relative displacements of two ends of the beam can be easily calculated. This formulation is powerful in the study of buckling of beams with various boundary conditions under compression, and can be used to calculate post-buckling equilibria of beams. Higher-order buckling modes of a long slender beam that have complex configurations are also studied using this formulation.

1. Introduction

Developing geometrically nonlinear models for beams and cables is an important subject in academic research [1,2] and engineering applications such as compliant mechanisms [3–5] and micro- and nano-electro-mechanical systems [6,7]. In compliant mechanisms, a pseudo-rigid-body (PRB) model for a beam is usually developed for their design and analysis. An analytical solution for the beam under end loads would be useful. Elliptic-integral solutions for a large-deflection Euler–Bernoulli cantilever beam with end loads have been given in Ref. [3]. While they are valid only for simple geometries and loadings, they have been widely used in developing various PRB models [4,5].

Formulations of beams and cables are generally considered to be different, but in practice, it is difficult to make such a distinction. Irvine [8] pointed out that a long slender beam behaves like a cable in the large; on the other hand, the small bending stiffness of a slack cable can be important in calculation of its equilibrium and dynamic response [9]. The study of equilibria of long slender beams is usually referred to as Kirchhoff–Love rod theory [1], and many researchers have made contributions to this area. Santillan et al. [10] theoretically and experimentally studied equilibria and stability of an elastic beam with two ends clamped together. Antman [2] provided a detailed derivation of equations of a beam using the rod theory, and Svetlitsky [11] discussed a similar topic from an engineering viewpoint. Kim and Chirikjian [12] studied equilibria and free vibration characteristics of a rod using a group-theoretical approach. Hodges [13] provided a systematic study on formulations of beams made of composite materials with arbitrary cross-sections, which can describe both slender and thick beams. Kumar [14] developed a generic model for partial delamination in composite beams using the finite element method (FEM). Romero et al. [15] developed a torsion-free non-linear beam model for beam-like slender structures whose cross sections can withstand only traction, shear, and bending.

In geometrically exact beam theories [16–18], spatial

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discretization of shear strains of cross sections can lead to some numerical issue called shear locking. Pai [19] pointed out that shear locking can be caused by combining the bending rotation and the shear rotation into one bending-shear rotation variable and reducing the order of the beam theory such as the Timoshenko beam theory. In numerical studies of nonlinear beams such as the FEM [20] and the absolute nodal coordinate formulation (ANCF) [21], bending and stretch deformations can be easily considered, but the shear deformation is usually neglected since it is so small, which leads to an Euler–Bernoulli beam model [22]. A new planar Rayleigh beam model was developed by Zhu et al. [9] using the slope angle of its centroid line. It uses much fewer number of generalized coordinates compared with the FEM. In practice, the magnitude of the shear strain can be larger than that of the stretch strain; a nonlinear model with stretch but without shear would not be appropriate in this case.

In this work, a planar beam is described by the slope angle and stretch strain of its centroid line, and shear strains of cross sections, so that bending, stretch, and shear deformations can be fully described, which extends the work in [9] where only the slope angle is considered. Static equations of equilibria are derived using the principle of virtual work and discretized using the finite difference method (FDM). The shear strain is expressed as a dependent variable, and the discretized model has no shear locking. When stretch and shear strains are small, an approximate solution can be obtained from a solution of the corresponding Euler–Bernoulli beam problem. Approximate stretch and shear strain distributions along the beam are determined from the slope angle distribution and compared with those from the FDM. Buckling loads of beams [23] are studied using the current formulation, and post-buckling equilibria of beams with various boundary conditions are accurately calculated. Higher-order buckling modes of a long slender beam that have complex configurations are also studied using the current formulation.

2. Nonlinear formulation of a planar beam under end forces and moments

Consider a planar beam of length L shown in Fig. 1. A cross-section of the beam can be described by an undeformed arc-length coordinate s along its centroid line, where $0 \le s \le L$. Suppose that cross-sections of the beam remain to be planar after deformation. General deformation of a planar beam includes bending and stretch deformations of its centroid line, and shear angles or strains of its cross-sections. The slope angle $\theta(s)$, the shear strain $\gamma(s)$, and the stretch strain $\epsilon(s)$ are used to describe the configuration of the deformed beam, and spatial coordinates of the particle on the centroid line at the cross-section corresponding to the arc-length coordinate s are

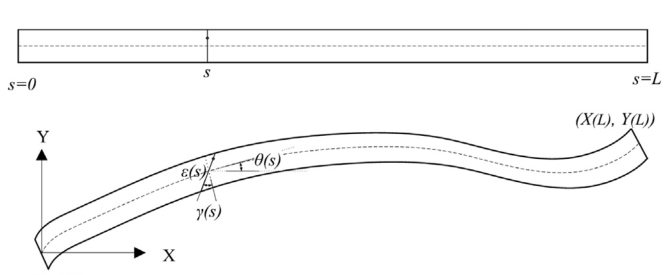

Fig. 1. Description of the configuration of a planar beam with large deformation using its slope angle $\theta(s)$, stretch strain $\epsilon(s)$, and shear angle or strain $\gamma(s)$.

$$X(s) = X_0 + \int_0^s \cos\theta(\tau)[1+\epsilon(\tau)]d\tau,$$
$$Y(s) = Y_0 + \int_0^s \sin\theta(\tau)[1+\epsilon(\tau)]d\tau, \tag{1}$$

where (X_0, Y_0) are coordinates of the particle on the centroid line at the cross-section corresponding to the arc-length coordinate $s=0$. The elastic potential energy of the beam is

$$\mathcal{P} = \frac{1}{2}\int_0^L \left\{ EA(s)\epsilon^2(s) + EI(s)\left[\frac{d\theta(s)}{ds} - \frac{d\gamma(s)}{ds}\right]^2 + \frac{GA(s)}{k(s)}\gamma^2(s)\right\} \times ds, \tag{2}$$

where E is Young's modulus, G is the shear modulus, $k(s)$ is the shear correction factor of the cross-section corresponding to the arc-length coordinate s, and $A(s)$ and $I(s)$ are the area and moment of inertia of the cross-section; the derivation is shown in Appendix A. Let

$$\phi(s) = \theta(s) - \gamma(s), \tag{3}$$

which is the bending angle of the cross-section corresponding to the arc-length coordinate s. The elastic potential energy in Eq. (2) becomes

$$\mathcal{P} = \frac{1}{2}\int_0^L \left[EA(s)\epsilon^2(s) + EI(s)\left(\frac{d\phi(s)}{ds}\right)^2 + \frac{GA(s)}{k(s)}\gamma^2(s)\right]ds, \tag{4}$$

and its variation is

$$\delta\mathcal{P} = EI(s)\phi'(s)\delta\phi(s)|_0^L - \int_0^L \left(EI(s)\phi'(s)\right)'\delta\phi(s)ds + \int_0^L EA(s)\epsilon(s)\delta\epsilon(s)ds + \int_0^L \frac{GA(s)}{k(s)}\gamma(s)\delta\gamma(s)ds, \tag{5}$$

where a prime denotes differentiation with respect to s. Suppose that there are external concentrated forces $\mathbf{F}_0 = \left(F_x^0,\ F_y^0\right)^T$ and $\mathbf{F}_1 = \left(F_x^1,\ F_y^1\right)^T$, and external concentrated moments m_0 and m_1 applied at two ends of the beam, and these forces and moments can be either prescribed or resultant forces and moments due to constraints. By Eqs. (1) and (3), one has

$$X(L) = X_0 + \int_0^L \left\{\cos\left[\phi(s)+\gamma(s)\right]\left[1+\epsilon(s)\right]\right\}ds,$$
$$Y(L) = Y_0 + \int_0^L \left\{\sin\left[\phi(s)+\gamma(s)\right]\left[1+\epsilon(s)\right]\right\}ds. \tag{6}$$

The virtual work done by the external forces and moments are

$$\delta\mathcal{W} = F_x^0\delta X_0 + F_y^0\delta Y_0 + m_0\delta\phi(0) + F_x^1\delta X(L) + F_y^1\delta Y(L) + m_1\delta\phi(L). \tag{7}$$

Substituting Eq. (6) into Eq. (7) yields

$$\begin{aligned}\delta\mathcal{W} = {} & \left(F_x^0 + F_x^1\right)\delta X_0 + \left(F_y^0 + F_y^1\right)\delta Y_0 + m_0\delta\phi(0) + m_1\delta\phi(L) \\ & + \int_0^L \left[F_y^1\cos\left(\phi+\gamma\right) - F_x^1\sin\left(\phi+\gamma\right)\right](1+\epsilon)\delta\phi ds \\ & + \int_0^L \left[F_y^1\cos\left(\phi+\gamma\right) - F_x^1\sin\left(\phi+\gamma\right)\right](1+\epsilon)\delta\gamma ds \\ & + \int_0^L \left[F_x^1\cos\left(\phi+\gamma\right) + F_y^1\sin\left(\phi+\gamma\right)\right]\delta\epsilon ds.\end{aligned} \tag{8}$$

Force and moment balance equations and static equations of equilibria of the beam are derived from the principle of virtual work $\delta\mathcal{P} = \delta\mathcal{W}$: