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Buckling and nonlinear dynamics of elastically coupled double-beam systems

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ABSTRACT

This paper deals with damped transverse vibrations of elastically coupled double-beam system under even compressive axial loading. Each beam is assumed to be elastic, extensible and supported at the ends. The related stationary problem is proved to admit both unimodal (only one eigenfunction is involved) and bimodal (two eigenfunctions are involved) buckled solutions, and their number depends on structural parameters and applied axial loads. The occurrence of a so complex structure of the steady states motivates a global analysis of the longtime dynamics. In this regard, we are able to prove the existence of a global regular attractor of solutions. When a finite set of stationary solutions occurs, it consists of the unstable manifolds connecting them.

1. Introduction

In this paper we investigate the properties of damped transverse vibrations and dynamical buckling of a coupled double-beam system under even compressive axial loading. The system models a sandwich structure with an elastic filler. It is composed of two equal WK-beams (according to the nonlinear model of Woinowsky-Krieger [37]), which are connected by linear springs and simply supported at the ends (see Fig. 1).

The in-plane dynamics is ruled by the following nonlinear system:

$$\begin{cases} \partial_{tt}u_1 + \delta\partial_{xxxx}u_1 + \nu\partial_t u_1 - \left(\ell + \gamma\,\|\partial_x u_1\|^2_{L^2(0,L)}\right)\partial_{xx}u_1 \\ \qquad + \kappa[u_1 - u_2] = f_1, \\ \partial_{tt}u_2 + \delta\partial_{xxxx}u_2 + \nu\partial_t u_2 - \left(\ell + \gamma\,\|\partial_x u_2\|^2_{L^2(0,L)}\right)\partial_{xx}u_2 \\ \qquad\qquad - \kappa[u_1 - u_2] = f_2, \end{cases} \tag{1}$$

where the unknown variables u_i: $[0, L] \times \mathbb{R}^+ \to \mathbb{R}$ $(i=1,2)$ represent the downward deflections in the vertical plane of the midline of the beams with respect to their reference configuration at rest. Both beams are hinged at their ends, so that

$$u_i(0, t) = u_i(L, t) = \partial_{xx}u_i(0, t) = \partial_{xx}u_i(L, t) = 0, \quad t \in [0, \infty), \quad i = 1, 2. \tag{2}$$

The unknown fields are required to satisfy the following initial conditions:

$$u_i(x, 0) = u_i^0(x), \quad \partial_t u_i(x, 0) = v_i^0(x), \quad x \in [0, L], \quad i = 1, 2, \tag{3}$$

where u_1^0, u_2^0, v_1^0 and v_2^0 are given functions which fulfill (2). The WK-beams are assumed to have equal length L and unitary mass. In the reference (natural) configuration they are straight and parallel, and their spacing is d. They are connected by linear elastic springs with common stiffness κ and free length d. Sources f_i, $i=1,2$, represent the given vertical load distributions. The positive constants δ and ν denote the flexural rigidity of the beams and the viscosity of the external environment, respectively. Finally, γ is a positive constant, whereas the parameter ℓ summarizes the effect of the axial force acting at one end of each beam and is positive when both beams are stretched, negative when compressed.

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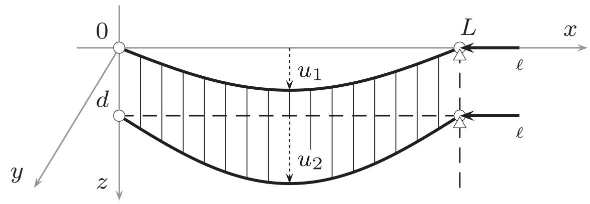

Fig. 1. In-plane oscillations of a double-beam sandwich system.

A derivation of the WK-beam model within a thermoelastic framework can be found in [18]. According to the modeling approach devised therein, the WK-beam equation is a simplification of the nonlinear von Kármán one-dimensional model, where longitudinal (horizontal in our case) displacements are condensed by integrating the corresponding equation in which longitudinal inertia is neglected. Unlike the usual Euler–Bernoulli linear theory, a nonlinear but uniform term accounting for extensibility of the beam is retained into the equation of the transversal vibration. We stress that all material constants in (1) are dimensionless (see [18] for details). In particular

$$\delta = \frac{h^2}{6L^2} > 0, \quad \ell = \frac{2D}{L} \in \mathbb{R},$$

where h and D are the thickness of the beam and its longitudinal displacement at the ends, respectively. As usual, both are assumed to be considerably shorter than the length L. Finally, we remark that the elongation of the coupling springs must take account of the horizontal displacements of their anchor points. Nevertheless, in (1) the strain in the springs is approached by the difference between the vertical displacements of the two beams. This may be accounted for assuming the maximum horizontal displacement, $|D|$, to be negligible if compared with the reference spacing of the beams, namely $|D| \ll d$.

System (1) may be also used to describe out-of-plane oscillations, both vertical and torsional, of a girder bridge where the road bed is modeled by an elastic rug connecting two lateral WK-beams (see Fig. 2). In this case, however, the lateral movements of the beams are neglected by the model.

Although all results obtained hereafter apply to both material models, for the sake of definiteness we shall refer to the former, only. In addition, due to the recasting of the problem into an abstract setting, we stress that the present analysis can be easily extended to (Berger) plate-type sandwich structures with hinged and normally loaded boundaries. In spite of a relatively wide literature concerning statics and dynamics of a single WK-beam (see e.g. [2,3,5,7,8,14,15,19,22,31,37] and references therein), we are not aware of analytic studies which consider the elastic coupling of two or more nonlinear beams of this type. On the other hand, mathematical models of sandwich beam-type and plate-type structures raised a wide interest in the literature, due to their relevance in many branches of modern civil, mechanical and aerospace engineering. In particular in the 80s the phenomenon of nonlinear buckling mode interaction stimulated much interest and has been investigated by many authors. In particular we recall the fundamental contribution by Budiansky [11] and the many technical papers by Sridharan (see for instance [4,32]). Recently, after

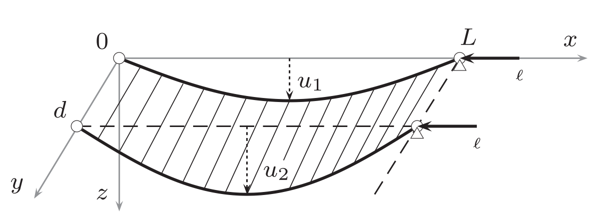

Fig. 2. Out-of-plane oscillations of a double-beam girder bridge.

some pioneer works (see, for instance, [23,29]) concerning interaction buckling between two beams, a lot of papers deal with mechanical properties of axially loaded and elastically connected linear double beam systems (see, for instance, [21,25,26, 35,36,38,39]).

The aim of this paper is to give a contribution on this subject by scrutinizing statics and dynamics of the initial boundary value problem (1)–(3). Its novelty and relevance relies on the complete characterization of the long time behavior, which emphasizes the different behavior of the nonlinear system (1) with respect to double-beam linear systems previously considered. It is worth noting that in a wider context concerning the localization of vibration modes and buckling patterns [24,27,30], nonlinearities play the same role than imperfections do in linear systems.

The occurrence of a very complex structure of the steady states motivates a global analysis of the longtime dynamics of system (1) (Sections 4 and 5). In this regard, due to the dissipative nature of the system ($\nu > 0$), we are able to prove the existence of a global regular attractor of solutions. In particular, when a finite set of stationary solutions occurs, the global attractor is given by the union of the unstable manifolds connecting them (Section 5.3).

2. Preliminary results

Introducing a suitable functional framework, we recast the original system (1) into an abstract setting. Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space, and let $A: \mathcal{D}(A) \Subset H \to H$ be a strictly positive selfadjoint operator, whose distinct eigenvalues and eigenfunctions are $\lambda_i > 0$ and ψ_i, $i \in \mathbb{N}$, respectively. For $\tau \in \mathbb{R}$, we introduce the Hilbert spaces

$$H_\tau = \mathcal{D}(A^{\tau/4}), \quad \langle u, v\rangle_\tau = \langle A^{\tau/4}u, A^{\tau/4}v\rangle, \quad \| u \|_\tau = \| A^{\tau/4}u \|.$$

The symbol $\langle\cdot,\cdot\rangle$ will also be used to denote the duality product between H_τ and its dual space $H_{-\tau}$. In particular, we have the compact embeddings $H_{\tau+1} \Subset H_\tau$, along with the generalized Poincaré inequalities

$$\lambda_1 \| w \|_\tau^4 \le \| w \|_{\tau+1}^4, \quad \forall\, w \in H_{\tau+1}, \tag{4}$$

and we define the family of product Hilbert spaces

$$\mathcal{H}_\tau = H_{\tau+2} \times H_\tau \times H_{\tau+2} \times H_\tau, \quad \tau \in [0, 2].$$

In all these notations the index τ is omitted when $\tau = 0$. Then, we state on $\mathcal{H}$ the following abstract Cauchy problem:

$$\begin{cases} \partial_{tt}u_1 + \delta A u_1 + \nu\partial_t u_1 + (\ell + \gamma\|u_1\|_1^2)A^{1/2}u_1 + \kappa(u_1 - u_2) = f_1, \\ \partial_{tt}u_2 + \delta A u_2 + \nu\partial_t u_2 + (\ell + \gamma\|u_2\|_1^2)A^{1/2}u_2 - \kappa(u_1 - u_2) = f_2, \end{cases} \tag{5}$$

$$(u_1(0), \partial_t u_1(0), u_2(0), \partial_t u_2(0)) = (u_1^0, v_1^0, u_2^0, v_2^0) \in \mathcal{H}. \tag{6}$$

The original problem (1)–(3) can be viewed as a special case of (5)–(6) by assuming $A = \partial_{xxxx}$ and $H = L^2(0, L)$. We stress that this abstract formulation cannot be applied when boundary conditions differ from (2) (for instance, if clamped–clamped or hinged–clamped ends are prescribed). Really, the original coupled system can be described by means of a single operator A only if the beams are assumed to be hinged at their ends. Afterwards, a weak solution of (5)–(6) will be denoted by $\sigma: \mathbb{R}^+ \to \mathcal{H}$,

$$\sigma(t) = (u_1(t), \partial_t u_1(t), u_2(t), \partial_t u_2(t)).$$

For further convenience, (5) may be rewritten as a system with a symmetric nonlinear coupling term independent of κ. Indeed, letting

$$w = \tfrac{1}{2}(u_1 + u_2), \quad v = \tfrac{1}{2}(u_1 - u_2), \quad f = \tfrac{1}{2}(f_1 + f_2), \quad g = \tfrac{1}{2}(f_1 - f_2),$$