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Exact analytical solutions for forced undamped Duffing oscillator

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ABSTRACT

The steady-state response of an undamped Duffing oscillator to periodic external forces is studied. The forcing functions are chosen such that the time course of the displacement can be described by exact analytical expressions. The displacement and forcing functions are governed by Jacobi elliptic functions and thus are periodic but generally non-harmonic. The parameter of the elliptic functions is deliberately chosen. For certain parameter choices, exact analytical expressions are found for the frequency–amplitude relation.

1. Introduction

A one-degree of freedom mechanical oscillator is considered in which the restoring force is a composition of linear and cubic functions of the displacement variable. When an external force is present and damping is absent, the equation of motion can be written

$$\ddot{v} + \omega^2 v + 2\kappa^2 v^3 = \frac{F}{M} \quad (1)$$

where $v = v(t)$ is the displacement variable, $F = F(t)$ is the external force, M is the moving mass of the oscillator, and a dot denotes derivation with respect to time, t. The system parameters, ω and κ, are non-negative real quantities. It is assumed that the restoring force corresponds to a hardening spring.

Eq. (1) is a particular case of the classical Duffing equation, which generally also contains a damping force term and in which the external force is usually assumed to vary harmonically with time or to be zero. The Duffing equation has extensively been studied in the past. Exact analytical solutions exist for Eq. (1) in the form of Jacobi elliptic functions provided $F(t) \equiv 0$ (free oscillation response) [1,2]. If damping is present or a harmonic external force is applied, only approximate analytical solutions exist. The harmonic balance method [3,4] and perturbation methods such as the Lindstedt-Poincaré [5], Krylov-Bogoliubov [6,7], or averaging [8,9] method are used to obtain approximate periodic solutions of such equations.

In the methods mentioned above, trigonometric approximations of the actual displacement response are normally used. In the case of strongly nonlinear oscillators, a qualitative improvement was observed when Jacobi elliptic functions are used instead of trigonometric functions for approximating the response and deriving approximate analytical solutions [4–9]. This observation suggests studying the response of such oscillators to external forces whose time course follows Jacobi elliptic functions.

This idea was formulated und used earlier [2,4,10–13]. Various types of Jacobi elliptic forcing functions were considered in [2] and it was shown that exact analytical solutions for the steady-state response to such forcing can be obtained. The parameter of the elliptic functions was left undetermined and amplitude–frequency relations were not fully developed. The idea was applied in [12,13] to purely cubic restoring force oscillators, which correspond to Eq. (1) without the second term. By choosing a particular Jacobi elliptic forcing function, derived from the free oscillation response and having a well-chosen parameter, an exact analytical solution was obtained for the steady-state response that facilitates the derivation of exact closed-form expressions for the amplitude–frequency relation of such oscillators.

The idea of applying Jacobi elliptic forcing functions to the study of the Duffing oscillator is revisited here. The particular case defined by Eq. (1) is considered. Special attention is given to choosing the parameter of the elliptic functions with regard to finding exact analytical expressions that relate the steady-state displacement response amplitude to the forcing frequency. In the following, first the free undamped oscillation response of the oscillator is clarified. On this basis, the forced steady-state oscillation is studied. The displacement function is chosen and a general expression for the corresponding external force function is derived. After considering various elliptic function parameter choices, preferred forms of the external force function are identified on which the further study is based.

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2. Free undamped oscillation response

The equation governing the free oscillation response of the oscillator under study here can be written

$$\ddot{v} + \omega^2 v + 2\kappa^2 v^3 = 0 \tag{2}$$

For solving Eq. (2), the trial displacement function

$$v(t) = \hat{v}\,\mathrm{sn}(\mu t \mid m) \tag{3}$$

is chosen, where $\hat{v}$ is the amplitude of the displacement, $\mathrm{sn}(\mu t|m)$ is a Jacobi elliptic function in parameter notation with the parameter m, and μ is a periodicity coefficient. Both μ and m are yet to be determined. The function $\mathrm{sn}(\mu t|m)$ appearing here corresponds to the initial condition $v(0) = 0$. When Eq. (3) is substituted into Eq. (2), the validity of the trial function is confirmed and expressions for μ and m are obtained:

$$\mu = \sqrt{\omega^2 + (\kappa\hat{v})^2} \geq 0 \tag{4}$$

$$m = -\frac{1}{1 + \left(\frac{\omega}{\kappa\hat{v}}\right)^2} \tag{5}$$

These expressions correspond to respective expressions in [2], where the motion is described by the Jacobi elliptic function $\mathrm{cn}(\mu t|m)$ that holds for the initial condition $\dot{v}(0) = 0$. In addition to the system parameters ω and κ, the displacement amplitude $\hat{v}$ appears in Eqs. (4) and (5). Hence the shape of the displacement function (3), which is controlled by m, and the frequency of the displacement, which is controlled by μ and m, also varies with $\hat{v}$. When defining the relative (dimensionless) displacement amplitude

$$V \stackrel{\text{def}}{=} \frac{\kappa\hat{v}}{\omega} \tag{6}$$

which can also be interpreted as a nonlinearity parameter, Eqs. (4) and (5) can be transformed into the following expressions in V:

$$\mu = \omega\sqrt{1 + V^2} \geq 0 \tag{7}$$

$$m = -\frac{1}{1 + V^{-2}} \tag{8}$$

The frequency of the free oscillation response described by Eq. (3) is

$$f = \frac{1}{T} = \frac{\mu}{4K(m)} = \frac{\mu}{2\pi}\varepsilon(m) \geq 0 \tag{9}$$

where T is the free oscillation period and $K(m)$ is the complete elliptic integral of the first kind [14]. $K(m)$ and the ratio $\varepsilon(m)$ defined as

$$\varepsilon(m) \stackrel{\text{def}}{=} \frac{K(0)}{K(m)} = \frac{\pi}{2}\cdot\frac{1}{K(m)} \tag{10}$$

depend on m. An explicit analytical relationship can thus be established for the free oscillation frequency–amplitude relation, that is, for f as a function of V or $\hat{v}$. When substituting Eqs. (7) and (8) into Eqs. (9) and (10) and referring to the free oscillation frequency f_{lin} of the associated linear oscillator (that has the same ω but in which $\kappa = 0$):

$$f_{\mathrm{lin}} = \frac{\omega}{2\pi} \geq 0 \tag{11}$$

and then defining the frequency ratio β:

$$\beta \stackrel{\text{def}}{=} \frac{f}{f_{\mathrm{lin}}} \geq 0 \tag{12}$$

the free oscillation frequency–amplitude relation is obtained in dimensionless form:

$$\beta = \sqrt{1 + V^2}\cdot\frac{\pi}{2}\left[K\left(\frac{-1}{1 + V^{-2}}\right)\right]^{-1} \geq 0 \tag{13}$$

It is apparent from Eq. (13) that β approaches 1, and f approaches f_{lin}, when V or, equivalently, $\hat{v}$ or κ, becomes small. Conversely, when ω approaches zero, Eq. (13) reduces to

$$\beta_{\mathrm{cubic}} = \pm\frac{V}{G} \Longleftrightarrow f_{\mathrm{cubic}} = \pm\frac{\kappa\hat{v}}{2\pi G} \tag{14}$$

in which f_{cubic} is the free oscillation frequency of the associated purely cubic restoring force oscillator (that has the same κ but in which $\omega = 0$) [15] and β_{cubic} is its dimensionless equivalent. The negative signs apply in case of negative V and $\hat{v}$. The quantity G appearing here is Gauss's constant defined as the arithmetic-geometric mean of 1 and $\sqrt{2}$ [16]:

$$G \stackrel{\text{def}}{=} \frac{1}{\mathrm{agm}(1,\sqrt{2})} = 0.83462\ldots \tag{15}$$

It is useful to recall the range of some of the terms:

$$0 \leq V^{-2} = \left(\frac{\omega}{\kappa\hat{v}}\right)^2 \leq \infty \Rightarrow -1 \leq m \leq 0 \tag{16}$$

$$\frac{\pi}{2}G \leq K(m) \leq \frac{\pi}{2} \Rightarrow \frac{1}{G} = 1.1981\ldots \geq \varepsilon(m) \geq 1 \tag{17}$$

Note that $\varepsilon(m)$ and $K(m)$ vary over a range of only about ±9%. Hence β is closely related to μ, or the square root term in Eq. (13), and its dependency on m, or the term related to K in Eq. (13), is less pronounced. Nevertheless, it seems impossible to invert relationship (13) to arrive at V as a function of β, that is, at an analytical expression for the free oscillation amplitude–frequency relation. However, this is unnecessary for plotting the corresponding graph, as shown in Fig. 1 later in this paper.

When evaluating Jacobi elliptic functions or elliptic integrals, some computer programs only accept parameters m for which $0 \leq m \leq 1$. In such a case, the terms $\mathrm{sn}(u|m)$ and $K(m)$ with $m < 0$, as they appear above and in the following, can be transformed into equivalent terms with $0 < m < 1$ [14].

3. Forced undamped steady-state oscillation response

In linear dynamics, exponential complex functions of the type $F(t) = \hat{F}e^{i\omega t}$ or corresponding trigonometric functions, such as $F(t) = \hat{F}\sin\omega t$, are normally used for representing the external force (harmonic loading), the main reason being mathematical advantages and not necessarily the resemblance of such functions with actually occurring loadings. In nonlinear dynamics, the advantages of using such functions are reduced. If a harmonic loading is nevertheless applied, the response of a nonlinear oscillator will not be of the same type. If an assumption to the contrary is made, closed-form solutions are possible, which, however, are approximate. If such an assumption is not made, only numerical solutions are possible.

It thus seems worthwhile to study the response of nonlinear oscillators to different types of loading that are periodic but non-harmonic and better fit the natural response tendency of the oscillator. In previous studies of the purely cubic restoring force