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Damaged hyperelastic membranes

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ABSTRACT

This paper deals with equilibrium problems in non-linear dissipative inelasticity of damaged membranes. The inelastic constitutive law is obtained by modifying the classical constitutive law for a hyperelastic isotropic material through a proper damage function, which allows to measure the effective stress and the dissipated energy. After making the constitutive modeling, the boundary-value problem is formulated for a damaged membrane subjected to biaxial loadings. The purpose of the analysis is to understand how behaves a membrane that, during the deformation process, experiences a progressively increasing damage. Equilibrium multiple branches of symmetric and asymmetric solutions, together to bifurcation points, are computed and it is shown how damage can alter these equilibrium paths with respect to the virgin undamaged case. In particular, the stress reductions caused by damage can give rise to transitions of the constitutive behavior from hardening type to the softening one. These changes can considerably affect the quality of the equilibrium solutions. Accordingly, the analysis is completed by assessing the stability of the solutions. For this aim, the stability analysis based on the energetic criterion is extended to damaged membranes.

1. Introduction

The non-linear behavior of the membranes has long been investigated. In one of its pioneering experiments, Treolar [1] stretched a square rubber membrane by means of uniformly distributed tensile forces acting on the four edges and having equal magnitude, observing the following somewhat surprising result: in some cases of high loading, the above doubly symmetric system of external forces may give rise to asymmetric stable deformations. Namely, the membrane may lose the square shape assuming the stable form of a parallelogram, characterized by two unequal principal stretches. In extreme cases, the difference between the two stretches approached 15%. Treolar [2] reported this symmetry-breaking phenomenon through experimental data, but without commenting on it in any way. For a critical interpretation of the experimental results of Treolar see the work of Batra et al. [3].

Intrinsic non-linearities of the equilibrium problem can therefore cause, even in a completely symmetric layout, an unexpected lack of symmetry on the deformation. From a theoretical point of view, the problem of the existence of asymmetric equilibrium solutions generated by symmetric loads has received attention only after a period of about 35 years. Kearsley [4], using the non-linear elasticity theory, analyzed membranes composed of incompressible Mooney–Rivlin material, finding both symmetric and asymmetric solutions. MacSithigh [5] studied the same problem using a minimum energy approach. For a general form of the incompressible stored energy function, Ogden [6] carried out a bifurcation analysis evaluating critical values of the tension at which bifurcation occurs and showing the nature of the deformation in the neighbourhood of these singularities. Another contribution to bifurcation analysis was provided by Haughton [7]. More recently, considering different choices of the stored energy function, explicit expressions for evaluating critical loads, bifurcation points and the global development of asymmetric solutions have been derived by Tarantino [8,9]. On the existence of asymmetric deformations due to symmetric boundary conditions it can also be seen the work of Fosdick and Royer-Carfagni [10].

In the last decades there has been, however, the emergence of more sophisticated theories which incorporate inelastic effects (see, e.g., [11–13]), exemplified by hysteresis, residual strain, thermal and viscoelastic effects, Mullin effects, frequency-dependent response and damage. All this is motivated by the need to more accurately model the material response of advanced rubber products in engineering applications, which often exhibit distinctly an inelastic behavior. This work is devoted to the analysis of damaged hyperelastic membranes in an isothermal and rate-independent context. We will confine ourselves to transformations slow enough so that the produced heat is transferred out to keep temperature variations unimportant. Likewise, the term rate-independent means that we are interested to phenomena with a time scale so slow that all rate effects (not only inertia) are neglected.

In the context of infinitesimal theory, the damage mechanics was introduced, about fifty years ago, by Kachanov [14] and then

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developed by Chaboche [15,16], Lemaitre [17–19] and Krajcinovic [20]. Damage mechanics was successfully applied to different classes of materials such as metals, composites and concretes (see, e.g., Voyiadjis et al. [21]). Concerning damaged rubber-like materials, it can be observed what follows. Mostly performing simple extension experiments, failure properties of natural rubbers have been investigated by Flory et al. [22] and by Smith [23,24]. At that time, several molecular theories of ultimate behavior of rubbers have also been developed and compared favorably with experiments [25,26]. However, these molecular models could not describe the accumulation of microscopic defects in the material. Thus, in the following years, many versions of damage approach with small elastic strain have been proposed and tested [27,28].

A more advanced and comprehensive approach, which includes large strains with and without damage, may be found in [11–13], where a new field theory for structured deformations, examining the effects of multiscale geometry accounting for the occurrence of disarrangements, has been introduced and examined. While the kinematics has been worked out in [29,30], dissipative and non-dissipative effects at submacroscopic levels, which include damage, have been examined in the three papers cited above.

Originally, damage mechanics was based on the definition of effective stress. Such a definition is motivated by the fact that, under loading, the material surface, on which internal forces apply, is decreasing because of the emergence of microdefects or microvoids.[1] These physical considerations are recalled in the next section, where the constitutive law for a damaged hyperelastic isotropic and compressible material is written in terms of Piola–Kirchhoff stress tensor, extending the effective stress concept and the hypothesis of strain equivalence [31]. In Section 3, the equilibrium problem of a membrane, experiencing a level of damage that increases gradually as enhances its deformation, is formulated. In detail, by varying the ultimate threshold of damage, it is investigated how damage may alter the equilibrium paths[2] of a virgin undamaged membrane. Then, formulated the boundary-value problem, the expressions governing the (symmetric and asymmetric) solutions, together to the conditions of occurrence of bifurcation, are derived. The study is completed in Section 4 where a stability analysis is proposed. For this purpose, the energy criterion, which states that a deformation is locally stable if it renders the total potential energy a minimum, is extended to the case of damaged membranes. In particular, four inequalities which, if fulfilled, ensure the stability of the solutions under each type of small perturbation, are obtained. Emphasis is placed to show how the damage may cause the loss of convexity of the total potential energy. In Section 5, considering the class of compressible Mooney–Rivlin materials, a numerical analysis is performed. The equilibrium branches, qualitatively more interesting, are shown by some diagrams and their stability is assessed.

2. Recalls on the damage theory

Let there be given a non-empty connected and bounded domain $\mathcal{B}$ of the three-dimensional Euclidean space $\mathfrak{E}$, whose boundary $\partial\mathcal{B}$ is Lipschitz-continuous. We identify the closure of such domain, $\overline{\mathcal{B}}$, as the body. The undeformed configuration $\overline{\mathcal{B}}$ of the body is assumed as the reference configuration, whereas the deformed configuration is given by the deformation $\boldsymbol{\varphi}:\overline{\mathcal{B}}\to\mathcal{V}$, that is a smooth enough, injective and orientation-preserving (in the sense that $\det\mathbf{F}>0$) vector field. Hereinafter $\mathbf{F}=\text{Grad}\boldsymbol{\varphi}$ denotes the deformation gradient and $\mathcal{V}$ the vector space associated with $\mathfrak{E}$.[3]

After these initial positions, we move on to describe the damage effects. Damage of materials is characterized by the progressive change in the microscopic internal structure of materials, which in turn leads to the deterioration of the mechanical properties. These material changes include the nucleation and growth of spatially distributed microcracks or microvoids under various loading conditions,[4] together with the mechanism of their coalescence to macrocracks or macrovoids.[5] Physically, when compared to the response of the virgin undamaged material, the existence of these microdefects results, on the one hand, in an *increase* of the stress level in the remaining effective material and, on the other hand, in a *decrease* of the stored energy function. An attractive theory is offered by *continuum damage mechanics* (CDM), which employs internal fields or rather damage variables for modeling, in an averaged sense, the local distribution of microdefects [20]. In the sequel, a restriction is made to isotropic damage states, assuming that microdefects have the orientation distributed uniformly in all directions. For isotropic damage states, a scalar damage variable usually produces satisfactory results (Davison et al. [42], Chaboche [43], Billardon and Moret-Bailly [44]). This scalar variable is denoted by d ($d=d_0\geq 0$ corresponds to the initial state and $d=d_c\leq 1$ to the final state. In particular, $d_c=1$ represents the complete local rupture).

The effective Cauchy stress tensor $\mathbf{T}_{\text{eff}}$, which acts on the damaged material, can be related to the usual Cauchy stress tensor $\mathbf{T}$ through the following expression (Chaboche [43], Murakami [45], Lemaitre and Chaboche [31]):

$$\mathbf{T}_{\text{eff}}=\frac{\mathbf{T}}{1-d}. \tag{1}$$

This relation is commonly applied with the hypothesis of *strain equivalence* [31]. Such an assumption states that the deformation behavior of a damaged material is expressed by the same constitutive law of the virgin undamaged material, in which the usual stress is replaced by the effective stress. Given the assumption of the strain equivalence, the deformation is affected by damage only in the form of the effective stress. This allows to write the *effective Piola–Kirchhoff stress tensor* $\mathbf{T}_{\text{Reff}}$ in a form similar to (1)

$$\mathbf{T}_{\text{Reff}}=\frac{\mathbf{T}_{\text{R}}}{1-d}. \tag{2}$$

With the previous assumptions, the stored energy function w can be considered as the product of the reducing factor $(1-d)$ by the stored energy function of the virgin material, denoted by w_0:

$$w(\mathbf{F},\ d)=(1-d)w_0(\mathbf{F}). \tag{3}$$

From (3) the (nominal) Piola–Kirchhoff stress tensor is obtained:

$$\mathbf{T}_{\text{R}}=(1-d)\frac{\partial w_0}{\partial\mathbf{F}}. \tag{4}$$

In the present purely mechanical setting, the thermodynamic framework makes use of two basic constitutive functions:

[1] Taking into account that the damage is modeled with the formation of a volume fraction of microvoids, the assumption of incompressibility may not be fully justified, although it may simplify the analysis.

[2] These paths are intended as the continuous sequence of equilibrium configurations that the body assumes, under the strain-control condition, by varying the external loads with continuity.

[3] In this paper, vector and vector-valued functions are represented by boldface minuscule letters, tensor and tensor-valued functions by majuscule letters. Differentiation with respect to time is denoted by a superimposed dot.

[4] A wide variety of (micro or macro) disarrangements are accounted for in [11–13].

[5] In non-linear elasticity, deformation and stress singular fields near the apex of macrocracks have been analyzed, for example, in [32–36] under static conditions, whereas the dynamical propagation of macrocracks has been studied in [37–39]. Also note that recently in fracture mechanics there has been a growing interest in the variational approach to these problems (see, e.g., [40,41]).