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# The motion of a piezoviscous fluid under a surface load ☆

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ABSTRACT

Using a variant of a spectral collocation method we numerically solve the problem of the motion of a highly viscous fluid with pressure dependent viscosity under a surface load, which is a problem relevant in many applications, in particular in geophysics and polymer melts processing. We compare the results with the results obtained by the classical Navier–Stokes fluid (constant viscosity). It turns out that for a realistic parameter values the two models give substantially different predictions concerning the motion of the free surface and the velocity and the pressure fields beneath the free surface.

As a byproduct of the effort to test the numerical scheme we obtain an analytical solution—for the classical Navier–Stokes fluid—of the surface load problem in a layer of finite depth.


## 1. Introduction

The fact that the material properties of fluids can depend on the pressure is known for a long time, see for example the well known pioneering experiments by Bridgman [1,2]. In many cases it is reasonable to model certain fluids as *incompressible* viscous or viscoelastic fluids with pressure dependent material moduli, see for example Bair et al. [3], Laun [4], Rajagopal and Szeri [5], Sedláček et al. [6], Sahaphol and Miura [7], Hausnerová et al. [8], Martínez-Boza et al. [9] and Kannan and Rajagopal [10] for applications ranging from polymer melts, lubricants and fuel oils to geomaterials.

A simple and frequently used mathematical model for some of these fluid type materials is the following generalisation of the Navier–Stokes model:

$$\mathbb{T} = -p\mathbb{I} + 2\mu(p)\mathbb{D}, \tag{1.1a}$$

where $\mathbb{T}$ denotes the Cauchy stress tensor, $p$ is the pressure, $\mathbb{D} =_{\text{def}} \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\top})$ is the symmetric part of the velocity gradient, and

$$\mu(p) = \mu_{\text{ref}} e^{\alpha(p - p_{\text{ref}})}, \tag{1.1b}$$

where $\mu_{\text{ref}}$ is the viscosity at the reference pressure $p_{\text{ref}}$. Such fluid is usually referred to as a piezoviscous fluid. Numerous papers have been devoted to analytical solutions of simple boundary value problems for model (1.1), see for example Denn [11], Hron et al. [12], Vasudevaiah and Rajagopal [13], Le Roux [14], Průša [15], Saccomandi and Vergori [16], Hron et al. [17], Kalogirou et al. [18], Rajagopal et al. [19–21], but the studies focused on numerical simulation of the behaviour of piezoviscous fluids in more complex settings are rather rare, see for example Gwynllyw et al. [22], Hron et al. [23], Lanzendörfer [24] or Chung and Vaidya [25].

In what follows we consider the motion of a viscous fluid with pressure dependent viscosity, that is of a fluid described by the constitutive relation (1.1), in a layer of a finite depth under a surface load, see Fig. 1 for the problem setting. The problem of the motion under a surface load appears in many applications, a prominent one being the problem of uplift of the Earth's crust after the melting of an ice sheet.[1] The problem has been studied—for the classical Navier–Stokes fluid and a layer of infinite depth—in the pioneering papers by Haskell [26,27], and although the setting used by Haskell [26,27] provides only a very oversimplified description of the postglacial rebound process, it still serves, in a sense, as a benchmark for testing the behaviour of material models in geophysics, see for example Mitrovica [28]. Interestingly, the influence of the pressure on the viscosity is of interest also for geomaterials, see for example Li et al. [29], which provides a motivation for the study of the motion of the piezoviscous fluid under a surface load.[2]

The paper is organised as follows. First we point out, see Section 2, two pleasing features concerning the investigation of the motion under a surface load in the context of geophysical applications,

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[1] In this case it makes sense to model the response of the Earth's crust as the response of a highly viscous *fluid*.

[2] The fully realistic geophysical setting would in many cases require to take into account also other effects such as temperature dependent viscosity and so forth. Here we want to focus only on the effects due to the pressure dependent viscosity, therefore we do not consider other relevant physical phenomena, and we basically stay in the oversimplified setting introduced by Haskell [26,27].

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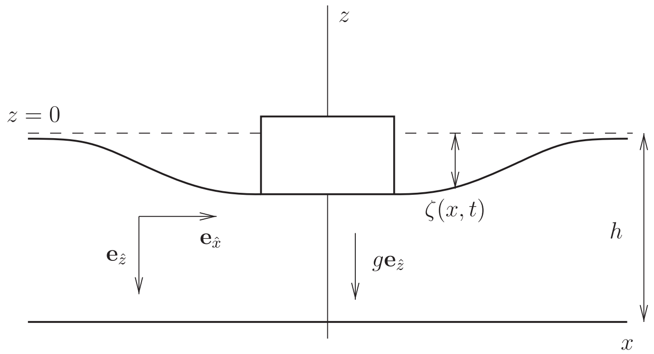


**Fig. 1.** Problem geometry.

namely the fact that the viscosity is very high, and that the deflection of the free surface is very small compared to the characteristic length scale of the problem. (The same holds for some polymer melts under a small surface load.) This means that the problem can be substantially simplified, see Section 2 for details, and instead of the full problem one can solve a simpler problem that provides a good approximate solution to the full problem.

The simplifications discussed in Section 2 are such that the only non-linearity that is treated in the simplified problem is the non-linearity in the constitutive relation, which allows one—in the case of the linear Navier–Stokes model—to solve the governing equations analytically, see Section 3. The analytical solution can be then used for testing the core component—the Stokes type problem solver—of the numerical scheme for the non-linear model (1.1).

In Section 4, we discuss a numerical scheme for solving the non-linear partial differential equations arising from the use of the constitutive relation (1.1). The scheme is based on a variant of a spectral collocation method, see for example Peyret [30] or Canuto et al. [31,32], which is a powerful method for solving partial differential equations in simple domains. Finally, in Section 5 we compare the results obtained by the classical Navier–Stokes fluid model and model (1.1). It turns out that the results are, for a realistic set of material parameters, significantly different.

## 2. Problem description

### 2.1. Full system of governing equations

Let us now discuss in detail the problem setting, see Fig. 1. Concerning the assumptions and simplifications we are going to make we closely follow Haskell [26,27]. The only departure from the original geometrical setting is the fact that we consider the motion under a surface load in a layer of a *finite* depth $h$. This is more realistic and in fact is necessary if we want to investigate the behaviour of the model (1.1), since in the layer of the infinite depth one would get infinite pressure and hence the viscosity in (1.1).

The problem of surface load is for simplicity studied as a two dimensional problem, there is no motion in the $\boldsymbol{e}_{\hat{y}}$ direction and all variables (velocity, pressure) are assumed to be independent of $y$ coordinate. (This is essentially tantamount to the claim that we are solving the problem for a load of infinite, meaning very large, extent in the $\boldsymbol{e}_{\hat{y}}$ direction.) The horizontal axis $z=0$ coincides with the surface of the unloaded layer, and the deflection of the free surface is described by the function $\zeta(x,t)$. The load $\sigma(x,t)$ is assumed to be symmetric with respect to $x=0$ axis, and the fluid is placed in the homogeneous gravitational field $\boldsymbol{b}=g\boldsymbol{e}_{\hat{z}}$.

The full system of governing equations for the motion of a homogeneous incompressible fluid in the domain $\Omega(t)=\{\boldsymbol{x}\in\mathbb{R}^2 | z\in(\zeta(x,t),h)\}$ reads, under the assumption of the absence of internal couples:

$$\rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=\mathrm{div}\,\mathbb{T}+\rho\boldsymbol{b}, \tag{2.1a}$$

$$\mathrm{div}\,\boldsymbol{v}=0, \tag{2.1b}$$

where $\boldsymbol{v}$ denotes the velocity and $\rho$ denotes the density. Symbol $\mathbb{T}$ denotes the Cauchy stress tensor which in the case of incompressible Navier–Stokes fluid and the piezoviscous fluid reads

$$\mathbb{T}=-p\mathbb{I}+2\mu\mathbb{D} \tag{2.2}$$

where $\mu=\mu_{\mathrm{ref}}$ or $\mu=\mu_{\mathrm{ref}}\mathrm{e}^{\alpha(p-p_{\mathrm{ref}})}$ respectively.

The boundary conditions on the bottom are the no-slip boundary and no-penetration boundary conditions:

$$\boldsymbol{v}|_{z=h}=\boldsymbol{0}, \tag{2.3}$$

and a traction type boundary condition on the free surface,

$$\mathbb{T}\boldsymbol{n}|_{z=\zeta(x,t)}=-\sigma\boldsymbol{n}, \tag{2.4}$$

where $\boldsymbol{n}$ denotes the unit outward normal to the surface $z=\zeta(x,t)$. Finally, the velocity and the stress fields are required to vanish for $x\to\pm\infty$.

One should be aware of the fact that the boundary condition (2.4) is only an approximation of a fully realistic setting, since this boundary condition implies that there is no interaction (such as friction) between the load and the underlying fluid, that is $\boldsymbol{t}\bullet\mathbb{T}\boldsymbol{n}|_{z=\zeta(x,t)}=0$, where $\boldsymbol{t}$ is the tangent to the surface $z=\zeta(x,t)$. Further, (2.4) also implies that the load always acts in the direction of the instantaneous normal to the (moving) free surface $z=\zeta(x,t)$ which is not necessarily true.

### 2.2. Simplified system of governing equations

Following Haskell [26,27] we will treat the problem in a quasi-static approximation, that is the left-hand side of (2.1a) is completely neglected, and the system reduces to

$$\boldsymbol{0}=-\nabla\tilde{p}+\mathrm{div}(2\mu\mathbb{D}), \tag{2.5a}$$

$$\mathrm{div}\,\boldsymbol{v}=0, \tag{2.5b}$$

where we have introduced the notation

$$\tilde{p}=_{\mathrm{def}}p-\rho gz. \tag{2.6}$$

The introduction of the modified pressure $\tilde{p}$ will lead, in the case of the Navier–Stokes fluid, to the elimination of the absolute term $\rho\boldsymbol{b}$ in (2.1a) and to the possibility of solving the linear homogeneous system (2.5a) for $\tilde{p}$ and $\boldsymbol{v}$ analytically, see Section 3 for details.

Concerning the boundary condition (2.4) we assume, following the seminal paper by Haskell [26,27], that the deflection of the free surface is small, and consequently (2.4) can be replaced by

$$\mathbb{T}|_{z=0}\boldsymbol{e}_{\hat{z}}-\rho gz|_{z=\zeta(x,t)}\boldsymbol{e}_{\hat{z}}=-\sigma\boldsymbol{e}_{\hat{z}} \tag{2.7}$$

The deflection of the free surface is therefore reflected only by the presence of the hydrostatic contribution to the spherical part of the Cauchy stress tensor and it is neglected elsewhere. (See Haskell [26,27] for details.) This approximation is the standard approximation in investigation of water waves, see for example the classical monograph by Lamb [33]. The simplified version of the boundary condition (2.4) therefore reads

$$\tilde{p}(x,0,t)+\rho g\zeta(x,t)-2\mu(p)\frac{\partial v_z}{\partial z}(x,0,t)=\sigma(x,t), \tag{2.8a}$$

$$\frac{\partial v_x}{\partial z}(x,0,t)+\frac{\partial v_z}{\partial x}(x,0,t)=0. \tag{2.8b}$$