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Multiple unstable equilibrium branches and non-linear dynamic buckling of shallow arches

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ABSTRACT

An arch under a suddenly-applied load will oscillate about its equilibrium position. If the suddenly-applied load is sufficiently large, the oscillation may reach a position on the unstable equilibrium branch of the arch, triggering its dynamic buckling. In many cases, arches are supported by other structural members or by elastic foundations which provide elastic types of rotational restraints to the ends of the arch. When the rotational end restraints of an arch are not equal to each other, the in-plane non-linear equilibrium path of the arch may have multiple unstable branches, which will influence the dynamic buckling of the arch significantly. This paper investigates effects of multiple unstable equilibrium branches on the non-linear in-plane dynamic buckling of a shallow circular arch under a suddenly-applied central concentrated load. Two methods based on the energy approach are used to derive the dynamic buckling loads. It is found that the number and magnitude of dynamic buckling loads are influenced significantly by the number of unstable equilibrium branches, by the stiffness of the unequal rotational end restraints, and by the included angle and slenderness ratio of the arch.

1. Introduction

It is known that if a shallow arch is subjected to a suddenly-applied load, the load will induce oscillation of the arch. When the load is sufficiently large, the arch may oscillate to its unstable equilibrium branch, triggering dynamic buckling of the arch. In practice, an arch may be supported by elastic foundations or by other structural members that provide elastic rotational restraints to its ends (Fig. 1). These rotational end restraints participate in the dynamic responses of the arch to the external dynamic loads and may influence its dynamic buckling behavior. It is known that when the stiffness of the end restraints is equal to each other, the non-linear equilibrium path of the arch under symmetric loading has a post-limit point unstable equilibrium branch or a post-bifurcation unstable branch [1,2]. In some cases, the configurations of the supporting structural members at both ends of an arch are not necessarily the same and so the rotational end restraints of the arch have unequal values of stiffness [3,4]. Arches with unequal rotational restraints under a uniform radial load only have one post-limit point unstable equilibrium branch [3]. However, when these arches are subjected to a central concentrated load, they have quite complicated non-linear equilibrium paths [4]. Owing to the influence of the rotational end restraints, the non-linear equilibrium path of an arch under a central concentrated load may have one, two or three post-limit point unstable equilibrium branches, and may even have looping postbuckling behavior [4]. In addition, the distance between the primary stable equilibrium branch and the unstable equilibrium branches is also influenced by the rotational end restraints significantly [4]. Such multiple postbuckling response problems of non-linear structures can also be solved by the generalized displacement control method as shown by Yang and Shieh [5].

Because dynamic buckling of an arch occurs when the oscillation caused by the suddenly-applied load reaches one of the unstable equilibrium branches [6,7], the number of unstable equilibrium branches and the distance between the primary stable and unstable branches are anticipated to play important parts in the non-linear dynamic buckling behavior of the arch. However, it is not known how to determine the dynamic buckling loads when an arch has the multiple unstable equilibrium branches and how the multiple unstable branches and unequal rotational end restraints influence the dynamic buckling of the arch. Hence, investigation is needed to solve these problems.

Under general loading, numerical and/or semi-analytical approaches are usually used to solve the differential equations of motion to determine the dynamic buckling load of structures [8–16]. However, when a structure such as a shallow arch whose non-linear equilibrium path has at least one unstable equilibrium branch is subjected to a suddenly-applied load, it is possible to use analytical approaches for its dynamic buckling analysis. Simitses

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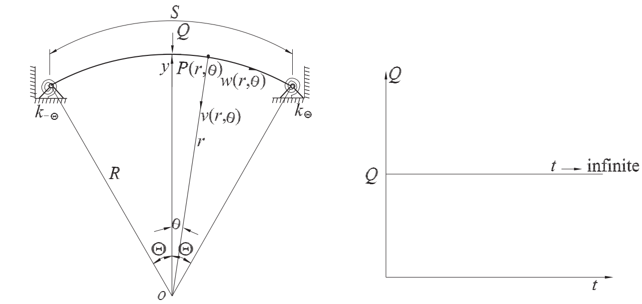

arch having unequal end rotational restraints sudden load of infinite duration

Fig. 1. Circular arch having rotational end restraints and suddenly-applied load.

[6] has shown that energy approaches can be used for dynamic buckling analyses of such structures under suddenly-applied loads, while Kounadis and his co-authors [17–24] have performed a number of excellent investigations and established criteria of non-linear dynamic buckling of various autonomous structural systems based on the energy-geometric approaches. The major merit of these energy approaches is that they are devoted to finding the criterion which allows the dynamic buckling load to be determined analytically without actually having to solve the differential equations of motion.

Energy approaches have also been used for the dynamic buckling analysis of shallow arches by several researchers. Simitses [6] derived approximate solutions for the lower and upper dynamic buckling loads for pin-ended and fixed shallow sinusoidal arches under suddenly-applied sine-wave loading. Levitas et al. [27] adopted Poincaré-like simple cell mapping to present a study of the global dynamic stability of a simply-supported shallow arch with a rectangular cross-section subjected to uniform constant lateral loading. Pi and Bradford [7,28,29] applied the energy approach to the dynamic buckling of pin-ended and fixed arches and obtained the analytical solutions for the dynamic buckling loads.

The energy approach is, therefore, used in this paper to study the effects of the multiple unstable equilibrium branches and unequal rotational end restraints on the dynamic buckling behavior of shallow pin-ended circular arches under a suddenly-applied central concentrated load of infinite duration (Fig. 1) and to use two complementary methods to derive the analytical solutions for the dynamic buckling loads of these arches.

2. Differential equations of motion

The rotational end restraints can be replaced by equivalent elastic rotational springs and the arch can be considered to be supported elastically at the ends by these rotational springs as shown in Fig. 1. Before dealing with the dynamic buckling behavior of a dissipative shallow arch, it is desirable to study the dynamic buckling behavior of its idealized undamped counterpart. Hence, the derivation of the differential equations of motion is based on the following assumptions: (1) The deformation of the arch satisfies the Euler–Bernoulli hypothesis, (2) the arch has a uniform cross-section, (3) the arch is slender, i.e. the dimensions of the cross-section are much smaller than the length and radius of the arch, (4) the lateral and torsional deformations of the arch are fully prevented, and (5) the arch is undamped. The solutions for the dynamic buckling of undamped arches provide a sound basis for investigating dynamic buckling of arches with small damping, which can be dealt with by using the energy method in association with a perturbation approach as shown in [6].

Based on these assumptions, the Lagrangian $\mathcal{L}$ of the undamped arch and load system can be expressed as

$$\mathcal{L} = \overline{T} - \overline{U}, \tag{1}$$

where $\overline{T}$ is the kinetic energy given by

$$\overline{T} = \frac{mA}{2}\int_{-\Theta}^{\Theta} R^3(\dot{\tilde{v}}^2 + \dot{\tilde{w}}^2)\,\mathrm{d}\theta, \tag{2}$$

where $\dot{()} = \partial()/\partial t$, t is the time, $\tilde{v} = v/R$, $\tilde{w} = w/R$, v and w are the radial and axial displacements, m is the mass density of the material, and $\overline{U}$ is the total potential energy of the arch and load system given by

$$\overline{U} = \int_{-\Theta}^{\Theta}\left[\frac{1}{2}\left(EAR\varepsilon_m^2 + EI_x\frac{\tilde{v}''^2}{R}\right) - QR\mathrm{Dirac}(\theta)\tilde{v}\right]\mathrm{d}\theta + \frac{1}{2}\sum_{i=\pm\Theta} k_i\tilde{v}_i'^2, \tag{3}$$

in which $k_i (i = \pm\Theta)$ is the stiffness of the rotational end restraints, E is Young's modulus, A is the area of the cross-section, I_x is the second moment of area of the cross-section about its major principal axis, $\varepsilon_{\rm m}$ is the membrane strain given by [6,17,25,27]

$$\varepsilon_m = \tilde{w}' - \tilde{v} + \frac{\tilde{v}'^2}{2}, \tag{4}$$

where $()' \equiv \partial()/\partial\theta$, and Dirac$(\theta)$ is the Dirac-delta function defined by

$$\mathrm{Dirac}(\theta) = \begin{cases} +\infty, & \theta = 0 \\ 0, & \theta \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\mathrm{Dirac}(\theta)\,\mathrm{d}\theta = 1, \tag{5}$$

and it has the property

$$\int_{-\infty}^{\infty}\mathrm{Dirac}(\theta)f(\theta)\,\mathrm{d}\theta = f(0). \tag{6}$$

The differential equations of motion can be derived from the Lagrangian $\mathcal{L}$ given by Eq. (3) using the Hamilton's principle, which can be stated as

$$\int_{t_1}^{t_2}\delta\mathcal{L}\,\mathrm{d}t = \int_{t_1}^{t_2}\delta(\overline{T} - \overline{U})\,\mathrm{d}t = 0 \quad \text{with } \delta\tilde{v} = 0,$$
$$\delta\tilde{w} = 0 \text{ at } t = t_1,\ t_2 \text{ for } -\Theta \le \theta \le \Theta \tag{7}$$

where t_1 and t_2 are arbitrary times.

Substituting Eqs. (2)–(4) into Eq. (7) and integrating it by parts leads to the differential equations of motion

$$(NR)' + mAR^3\ddot{\tilde{w}} = 0 \tag{8}$$

in the axial direction, and

$$-\frac{EI_x\tilde{v}^{iv}}{R} - (NR\tilde{v}')' - NR + \mathrm{Dirac}(\theta)QR - mAR^3\ddot{\tilde{v}} = 0 \tag{9}$$

in the radial direction; and leads to the static boundary conditions as

$$2\Theta\alpha_i\tilde{v}_i'' \pm \tilde{v}_i' = 0 \quad \text{with } i = \pm\Theta \tag{10}$$

where N is the axial compressive force and α_i is the relative flexibility of the elastic rotational end restraints defined by

$$\alpha_i = \frac{EI_x}{k_iS} \quad \text{with } i = \pm\Theta \tag{11}$$

with S being the length of the arch.

In addition, the essential kinematic boundary conditions are

$$\tilde{v} = 0 \quad \text{and} \quad \tilde{w} = 0 \text{ at } \theta = \pm\Theta. \tag{12}$$

When the arch is assumed to be at rest before the application of the sudden central concentrated radial load, the initial conditions are

$$\tilde{v} = \tilde{w} = 0 \quad \text{and} \quad \dot{\tilde{v}} = \dot{\tilde{w}} = 0 \text{ at } t = 0. \tag{13}$$