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On the modeling of the non-linear response of soft elastic bodies

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ABSTRACT

In this short note we articulate the need for a new approach to develop constitutive models for the non-linear response of materials wherein one is interested in describing the Cauchy–Green stretch as a non-linear function of the Cauchy stress, with the relationship not in general being invertible. Such a material is neither Cauchy nor Green elastic. The new class of materials has several advantages over classical elastic bodies. When linearized under the assumption that the displacement gradient be small, the classical theory leads unerringly to the classical linearized model for elastic response, while the current theory would allow for the possibility that the linearized strain be a non-linear function of the stress. Such bodies also exhibit a very desirable property when viewed within the context of constraints. One does not need to introduce a Lagrange multiplier as is usually done in the classical approach to incompressibility and the models are also more suitable when considering nearly incompressible materials. The class of materials considered in this paper belongs to a new class of implicit elastic bodies introduced by Rajagopal [19,20]. We show how such a model can be used to interpret the data for an experiment on rubber by Penn [18].

1. Introduction

A material such as rubber is capable of sustaining large deformations wherein the relationship between the strain and stress is non-linear, and since the dissipation is negligible such materials are modeled as non-linear elastic bodies. The classical approach to modeling such models is to describe them as Cauchy elastic [5,6] bodies or Green elastic bodies[1] (see Truesdell [29] for a definition of such elastic bodies). Recently, a much larger class of models have been proposed to model the response of such materials by Rajagopal (see [19,20]) which includes the classical Cauchy and Green elastic bodies as a subclass but it also includes another subclass, models wherein an appropriate kinematical measure is expressed as a function of the Cauchy stress. *Such an approach has several distinct advantages over the classical model: first, when such models are linearized under the assumption that the displacement gradient be small, they lead to models wherein the linearized strain is a non-linear function of the stress; second, within the context of such new models there is no need to introduce a Lagrange multiplier when dealing with a constraint such as incompressibility; third, such models are more in keeping with the notion of causality in classical Newtonian mechanics wherein force (and consequently the surface traction and thus the stress) is viewed as the cause and the kinematics is viewed as the effect.*

With regard to the first advantage, we recall that linearization based on the displacement gradient being small within the context of the classical model leads inexorably to the classical linearized elastic model. This presents an interesting predicament to describing response wherein the relationship between the strain and the stress is non-linear even within the context of small strains. Of course, all sorts of ad hoc models wherein stress is expressed as a non-linear function of the linearized strain have been used, but these models have no proper basis and are self-contradictory as non-linearities in the strain are ignored in the first place in defining the linearized strain.[2] The dilemma of

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[1] Green elastic bodies [13,14], also referred to as hyperelastic bodies, can be considered as a subclass of Cauchy elastic bodies, wherein the stress in such materials is derivable from a potential. Green [14] observed that an elastic body whose stress is not derivable from a potential would give rise to a perpetual motion machine. Recently, Carroll [4] showed that a Cauchy elastic body that is not Green elastic is indeed an infinite source of energy. However, it is possible that there could be elastic bodies that are not Cauchy elastic (nor Green elastic) which are defined through implicit constitutive relations between the Cauchy–Green tensor and the stress, and the stress in such bodies need not necessarily be derivable from a potential. However, within this more general class of bodies, there are counterparts of Green elastic bodies in that the strain is derivable from a complimentary stress potential. This study is devoted to the behavior of bodies described by implicit constitutive relations between the Cauchy–Green tensor and the stress that are neither Cauchy elastic nor Green elastic.

[2] The new class of models which allows for the linearized strain to depend non-linearly on the stress allows one to tackle one of the most difficult problems in solid mechanics in a consistent manner. The manner in which the problem of fracture has been treated within the framework of linearized elasticity is totally inconsistent. At the tip of a crack, the linearized theory of elasticity predicts that

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needing a model wherein the stress and the linearized strain are related in a non-linear manner can be overcome if one could have the possibility of elastic bodies that are more general than Cauchy elastic or Green elastic bodies which when linearized would lead to models wherein the stress and linearized strain are related in a non-linear manner. Since one cannot have consistency if one linearizes the non-linear strain but allows for a non-linear function of the linearized strain, we immediately realize that a non-linear relationship between the linearized strain and stress is only possible by allowing a relationship in which the linearized strain appears linearly, but the stress appears in a non-linear fashion. This is precisely what is achievable with the new class of elastic bodies introduced recently by Rajagopal and co-workers (see [19–22]). The advantages with regard to not having to introduce a Lagrange multiplier is that we do not have to deal with a part to the stress that is indeterminate. Constraints also become a part of the constitutive specification.

Rajagopal [19] introduced implicit constitutive relations between the stress and the Cauchy–Green stretch to describe the response of elastic bodies. The implicit class of bodies introduced by Rajagopal includes Cauchy elastic bodies as a special subclass. Later, Rajagopal and Srinivasa [24,25] provided a thermodynamic basis for such elastic bodies and Bustamante and Rajagopal [2,3] developed implicit constitutive relations to describe the response of electroelastic bodies. Recently, Freed [12] has shown that such models can be used to describe the response of soft solids.

We fit the experimental data of Penn [18] with the aid of the new class of models. This data could also be fit with the classical Green elastic model. However, our aim is to show that the new class of models, which has several advantageous features over the classical Cauchy and Green elastic bodies, can be used to fit the data as well as the classical model.

The organization of the paper is as follows. In the next Section we introduce the kinematics and develop the constitutive model. In Section 3, we fit the experimental data obtained by Penn [18] with a specific model that belongs to the class derived in the previous section.

2. Constitutive relations

Let $\mathbf{x}$ denote the current position of a particle[3] which is at $\mathbf{X}$ in a stress-free reference configuration. Let $\mathbf{x}=\boldsymbol{\chi}(\mathbf{X},t)$ denote the motion[4] of a particle and let us denote the displacement by

$$\mathbf{u} := \mathbf{x}-\mathbf{X}. \tag{2.1}$$

The displacement gradients $\partial\mathbf{u}/\partial\mathbf{X}$ and $\partial\mathbf{u}/\partial\mathbf{x}$ are given by

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \nabla_{\mathbf{X}}\mathbf{u} = \mathbf{F}-\mathbf{1}, \tag{2.2}$$

and

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \nabla_{\mathbf{x}}\mathbf{u} = \mathbf{1}-\mathbf{F}^{-1}, \tag{2.3}$$

where $\mathbf{F}$ is the deformation gradient defined through

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}. \tag{2.4}$$

The Cauchy–Green stretch tensors $\mathbf{B}$ and $\mathbf{C}$ are defined through

$$\mathbf{B} := \mathbf{F}\mathbf{F}^T, \quad \mathbf{C} := \mathbf{F}^T\mathbf{F}. \tag{2.5}$$

The Green–St. Venant strain $\mathbf{E}$ and the Almansi–Hamel strain $\mathbf{e}$ are defined through

$$\mathbf{E} := \frac{1}{2}(\mathbf{C}-\mathbf{1}), \quad \mathbf{e} := \frac{1}{2}(\mathbf{1}-\mathbf{B}^{-1}). \tag{2.8}$$

The linearized strain $\boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right]. \tag{2.9}$$

The above kinematical definitions suffice for our purpose.

We recall that a body is said to be Cauchy elastic if the Cauchy stress is related to the deformation gradient through

$$\mathbf{T} = \mathbf{g}(\mathbf{F}), \tag{2.11}$$

for all non-singular deformation gradients. Frame indifference places restrictions on the constitutive Eq. (2.11) and it follows that the stress has to be expressed as

$$\mathbf{T} = \mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{R}^{\mathbf{T}}. \tag{2.12}$$

where $\mathbf{R}$ and $\mathbf{U}$ are the rotation and the stretch that appear in the polar decomposition $\mathbf{F}=\mathbf{R}\mathbf{U}$.

On appealing to standard methods used in continuum mechanics, in the case of a general compressible homogeneous isotropic Cauchy elastic body (see Truesdell and Noll [30]) the Cauchy stress is given by:

$$\mathbf{T} = \delta_1\mathbf{I} + \delta_2\mathbf{B} + \delta_3\mathbf{B}^2, \tag{2.13}$$

where δ_i, $i=1, 2, 3$ depends on ρ, $I_1=tr\mathbf{B}$, $I_2=(1/2)\{(tr\mathbf{B})^2-tr\mathbf{B}^2\}$, and $I_3=\det\mathbf{B}$, or equivalently by ρ, $tr\mathbf{B}$, $tr\mathbf{B}^2$ $tr\mathbf{B}^3$. In fact any integrity basis can be used and we will see that while the integrity basis documented above might be suitable for theoretical calculations, it gives rise to serious problems with regard to data reduction.

In the case of an isotropic homogeneous non-linear elastic body, the stress is given by

$$\mathbf{T} = -p\mathbf{I} + \delta_2\mathbf{B} + \delta_3\mathbf{B}^2, \tag{2.14}$$

where $-p\mathbf{I}$ denotes the indeterminate part of the stress due to the constraint of incompressibility, and δ_i, $i=1, 2$, depends on ρ, $tr\mathbf{B}$, $(1/2)\{(tr\mathbf{B})^2-tr\mathbf{B}^2\}$, Since the material is incompressible, it can undergo only isochoric motions and hence $\det\mathbf{B}=1$. It is important to recognize that p is not the "pressure" in the solid where usually the term "pressure" is used to signify the mean normal stress in the body.

Recently, Criscione and co-workers (see [7–11]) have pointed out that using an integrity basis that consists in the principal invariants of the tensor $\mathbf{B}$ or the integrity basis that is given in the line following Eq. (2.12) gives rise to serious difficulties when it comes to experimental data reduction. This is because of what Criscione and co-workers refer to as the "collinearity" of the integrity basis.[5] In a series of papers, Criscione and co-workers

(*footnote continued*)
the strain has a singularity and grows like $1/\sqrt{r}$, where r is the radial distance from the tip of the crack. One might try to absolve this inconsistency by stating that the response close to the tip of the crack is dissipative, and numerous ad hoc fixes based on the response being inelastic have been proposed. However, in a brittle material, at sufficiently low temperature, one expects cracks to propagate without significant dissipation. The new class of materials allows one to show that strains can remain bounded and below a value that can be fixed a priori to be small thereby not leading to any inconsistency in the analysis (see Rajagopal and Walton [26], Kulvait et al. [16], Ortiz et al., [17], Bulicek et al. [1]).

[3] We shall not concern ourselves with a rigorous definition of what is meant by a body, placer, configuration, etc. The interested reader can refer to the book by Truesdell and Noll [30].

[4] As the form of the motion depends on the choice of the reference configuration, if one wanted to be precise one would identify this fact by indexing the motion to reflect this fact. In the interest of simplicity, we shall not do so here.

[5] The difficulty that arises in the data reduction can be best understood with a very simple example. While $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $\mathbf{i}, \mathbf{i}+10^n\mathbf{j}, \mathbf{k}$ also form a basis, in the latter basis certain small changes in a directed line segment get distorted tremendously or a large change get trivialized, depending on the value of n, how large and whether positive or negative, leading to an erroneous interpretation of the results (see [23]).