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Growth-induced axial buckling of a slender elastic filament embedded in an isotropic elastic matrix

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ABSTRACT

We investigate the problem of an axially loaded, isotropic, slender cylinder embedded in a soft, isotropic, outer elastic matrix. The cylinder undergoes uniform axial growth, whilst both the cylinder and the surrounding elastic matrix are confined between two rigid plates, so that this growth results in axial compression of the cylinder. We use two different modelling approaches to estimate the critical axial growth (that is, the amount of axial growth the cylinder is able to sustain before it buckles) and buckling wavelength of the cylinder. The first approach treats the filament and surrounding matrix as a single 3-dimensional elastic body undergoing large deformations, whilst the second approach treats the filament as a planar, elastic rod embedded in an infinite elastic foundation. By comparing the results of these two approaches, we obtain an estimate of the foundation modulus parameter, which characterises the strength of the foundation, in terms of the geometric and material properties of the system.

1. Introduction

Cylindrical structures are ubiquitous throughout the biological world. Examples can be found across a broad range of length scales, from microtubules within the cell to macroscopic plant stems. Every type of structure fulfils a key role within its relevant environment, such as oxygen and nutrient transport provided by airways and arteries, or signal transmission carried out by axons. Biological structures are often subject to differential growth, a process whereby different regions of tissue within the same body grow at different rates. This gives rise to residual stresses that exist in the absence of any applied traction, and have important effects on the overall mechanical behaviour of a material body [7,8]. Should the stresses within a body become sufficiently large, material failure can occur and this may be manifested in various ways, including fracture, cracking, plastic yield or buckling.

Rod and beam-like structures are often embedded within another material. Examples include plant roots growing in soil, microtubules embedded within the cytoplasm, and blood vessels surrounded by body tissue. A simple modelling approach that can be applied to each of these scenarios is to treat the surrounding material as an elastic foundation. Many studies of such embedded slender structures make use of the classical Euler beam theory [4] in order to study the onset of buckling under the action of a compressive force. Murmu and Pradhan [17] used Timoshenko beam theory (an extension of Euler beam theory) to investigate the effect of various types of elastic foundation on the critical longitudinal buckling stress (*i.e.* the compressive stress at which the tube buckles) of embedded single-walled carbon nanotubes. Their analysis made use of a foundation modulus parameter, k_f, which characterises the amount of tranverse reinforcement of the foundation acting on the tube [24,23,5]. They found that the critical buckling pressure increases with k_f, although no information is given on how k_f can be measured for a specific material. Brangwynne et al. performed buckling experiments on microtubules. They applied axial loads to microtubules under two different circumstances. First, when excised from the cell they observed that under small axial loads, microtubules exhibited Euler-type buckling [3]. However, when embedded within the cytoplasm they were able to withstand a greater compressive force (imposed by applying a normal force at the point where the microtubule meets the cell membrane) before the onset of buckling. Moreover, once buckling did occur, the observed wavelength was shorter than when isolated microtubules were considered. The authors also carried out a theoretical analysis by treating a microtubule as a cylindrical, inextensible beam embedded within an elastic foundation. They proposed that the observed buckling mode is that which minimises the sum of the beam's bending energy and the energy required to transversely displace the surrounding cytoplasm. Their analysis uses the parameter α to characterise the transverse reinforcement of the cytoplasm, and they estimate this parameter in terms of the radius of the rod, a, the Young's modulus of the surrounding matrix, G, and the characteristic buckling length scale of the structure, l as follows:

$$\alpha_{e1} = \frac{4\pi G}{\log\left(\frac{l}{a}\right)}. \tag{1}$$

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However, this approach is problematic, since the wavelength, l, is not known *a priori*. Therefore this estimate cannot be used in any predictive way. Furthermore, its derivation remains mysterious. An alternative estimate of this parameter where the wavelength is replaced by the length L of the beam was presented in [20], and given by

$$\alpha_{e2} = \frac{4\pi G}{\log\left(\dfrac{L}{a}\right)}. \tag{2}$$

As we will see, this estimate cannot be correct either as the buckling properties of a long filament in a matrix are essentially independent of the length of the filament. By comparing the exact buckling properties of a cylinder under axial load in an infinite matrix with the properties of a Kirchhoff rod on an elastic foundation we derive a new estimate for the foundation modulus parameter.

1.1. The problem

We consider the problem of an isotropic, slender, elastic filament embedded in an isotropic, elastic matrix. The filament and matrix are constrained in the axial direction by two flat, rigid plates and the filament undergoes uniform axial growth, so that it becomes axially compressed. At a critical growth, the filament buckles. We model this problem first by considering volumetric growth in a 3-dimensional, non-linear elastic body. This approach has been used in a number of previous studies of axial and circumferential buckling of hollow multi-layered cylinders with finite radii [10,21,22,14], and exploits the idea of multiplicative decomposition of the deformation tensor into two components: stress-free growth and elastic response. The framework was originally proposed by Rodriguez et al. [19], and has since been widely incorporated into many models of volumetric growth, *e.g.* [14]. For the case of a neo-Hookean filament and neo-Hookean matrix we are able to estimate the critical growth value analytically via the use of the WKB method. Second, we consider a rod-theory formulation that models the system as a 2-dimensional elastic beam embedded in a Winkler foundation. Like previous rod-theory models, this approach makes use of a foundation modulus parameter. Each approach provides us with an estimate of the critical axial growth of the filament and the wavelength of the resulting buckled state. The goal of this paper is to fully describe this buckling instability and obtain an estimate of this parameter by comparing the results of two approaches. This foundation modulus parameter is directly related to geometric and material properties of the system. Further, the generalisation to a Mooney–Rivlin matrix does not alter this estimate.

2. 3-D elasticity approach

2.1. General setup

We follow closely the method outlined in [1]. Consider an elastic body occupying a reference (stress-free) configuration, $\mathcal{B}_0$, defined by co-ordinates $\mathbf{X}$. The body is subject to a deformation, χ, so that its new configuration, $\mathcal{B}_f$ (which we shall refer to as the current configuration), is defined by co-ordinates $\mathbf{x} = \chi(\mathbf{X})$. Let $\mathbf{F} = d\chi/d\mathbf{X} = \mathrm{Grad}(\chi)$ be the geometric deformation tensor.

As described in the Introduction, the effects of growth are incorporated into the model via multiplicative decomposition of the deformation gradient tensor, so that $\mathbf{F}=\mathbf{AG}$, where $\mathbf{G}$ represents local stress-free growth and $\mathbf{A}$ represents the elastic response. Since we will restrict our attention to semi-inverse problems where cylinders are mapped onto other cylinders, a full description of the kinematics of growth is not required.

We assume that the body is incompressible, which implies that only isochoric deformations are possible, that is

$$\det \mathbf{A} = 1. \tag{3}$$

We also assume that the body is hyperelastic, that is there exists a strain-energy density, W, such that

$$\mathbf{T} = \mathbf{A}\frac{\partial W}{\partial \mathbf{A}} - p\mathbb{1}, \tag{4}$$

where $\mathbf{T}$ is the Cauchy stress tensor and p is a Lagrange multiplier enforcing the incompressibility constraint. In the absence of body forces, the equation of static mechanical equilibrium can be written as

$$\mathrm{div}\ \mathbf{T} = \mathbf{0}. \tag{5}$$

For our problem, a natural choice for boundary conditions is to prescribe the deformation on some part of the boundary and impose a pressure, P, acting in the normal direction, $\hat{\mathbf{n}}$, on the rest of the boundary:

$$\mathbf{T}.\hat{\mathbf{n}} = -P\hat{\mathbf{n}}. \tag{6}$$

2.2. Incremental deformations

We now investigate the stability of solutions of (5). We do so by introducing a small perturbation to the finite deformation. This perturbation is an incremental deformation, that belongs to a wider class of deformation with no prescribed symmetry, and is defined by

$$\chi(\mathbf{X}) = \chi^{(0)}(\mathbf{X}) + \epsilon\chi^{(1)}(\mathbf{X}), \tag{7}$$

where $0 < \epsilon \ll 1$ characterises the size of the perturbation. Accordingly, we define

$$\mathbf{F} = (\mathbb{1} + \epsilon\mathbf{F}^{(1)})\mathbf{F}^{(0)}, \quad \mathbf{A} = (\mathbb{1} + \epsilon\mathbf{A}^{(1)})\mathbf{A}^{(0)}. \tag{8}$$

The incremental equations and boundary conditions are formulated in terms of the current configuration. We expand the Cauchy stress tensor as follows

$$\mathbf{T} = \mathbf{T}^{(0)} + \epsilon\mathbf{T}^{(1)} + O(\epsilon^2). \tag{9}$$

Substitution of (9) into (4), gives

$$\mathbf{T}^{(0)} = \mathbf{A}^{(0)}\frac{\partial W^{(0)}}{\partial \mathbf{A}^{(0)}}, \tag{10}$$

$$\mathbf{T}^{(1)} = \mathcal{L} : \mathbf{A}^{(1)} + \mathbf{A}^{(1)}\mathbf{A}^{(0)}W_A^{(0)} - p^{(1)}\mathbb{1}, \tag{11}$$

where $p = p^{(0)} + \epsilon p^{(1)}$ and $\mathcal{L}$ is the fourth-order tensor given by

$$\mathcal{L} : \mathbf{A}^{(1)} = \mathbf{A}^{(0)}(W_{AA}^{(0)} : \mathbf{A}^{(1)})\mathbf{A}^{(0)}. \tag{12}$$

Here, $W_A^{(0)}$ and $W_{AA}^{(0)}$ are respectively the first and second derivatives of W with respect to $\mathbf{A}$ evaluated at $\mathbf{A}^{(0)}$. Explicitly, the nonzero components of $\mathcal{L}$ are given by [18]

$$\left.\begin{aligned}
&\mathcal{L}_{iijj} = \mathcal{L}_{jjii} = \alpha_i\alpha_j\frac{\partial^2 W}{\partial\alpha_i\partial\alpha_j}, \\
&\mathcal{L}_{ijij} = \alpha_i^2\frac{\alpha_i\dfrac{\partial W}{\partial\alpha_i} - \alpha_j\dfrac{\partial W}{\partial\alpha_j}}{\alpha_i^2 - \alpha_j^2}, && i \neq j, \alpha_i \neq \alpha_j, \\
&\mathcal{L}_{ijij} = \frac{\mathcal{L}_{iiii} - \mathcal{L}_{iijj} + \alpha_i\dfrac{\partial W}{\partial\alpha_i}}{2}, && i \neq j, \alpha_i = \alpha_j, \\
&\mathcal{L}_{ijij} - \mathcal{L}_{ijji} = \mathcal{L}_{ijij} - \mathcal{L}_{jiij} = \alpha_i\frac{\partial W}{\partial\alpha_i}, && i \neq j,
\end{aligned}\right\} \tag{13}$$

where α_i are the principal values of $\mathbf{A}^{(0)}$. The equilibrium equations are then given by

$$\mathrm{div}(\mathbf{T}^{(0)}) = \mathbf{0}, \tag{14}$$