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Reconstruction of the velocities and pressures for a viscous fluid in a hollow cylinder

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Abstract

We find the continuous velocities and pressures at all points of an incompressible viscous fluid flowing in a hollow cylinder from the given velocities at the points of some closed curves on the inner sides of the cylinder and the given pressures at two points on the inner sides of the cylinder. This solution can be named the interpolation solution. The solution is reduced to a succession of plane boundary problems for the elliptic differential equations. The method of solution is programmable and applicable for problems of piping, for example, for modelling a blood flow in a vessel with a fibrin on its inner surface. This method is illustrated by application to flow in a hollow circular cylinder with a rotation on the central level.

The interpolation solution is the basis for the method of restoration of continuous velocities and pressures in the flow of an incompressible viscous fluid from the given velocities at a finite number of points on the inner surface of the cylinder and given pressures at two points on the same surface.

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1. Introduction

We examine slow stable flow at low Reynolds number for an incompressible viscous fluid in a hollow cylinder with some interference on the inner sides of the cylinder which influence the velocities. An example of such a flow is blood flow in a vessel with a fibrin on the inner surface. Given the velocities at the points of $n+1$ closed curves on the inner side of the cylinder and the pressures at two points on the inner sides of the cylinder, it should be possible to find the velocities and pressures at any point inside the cylinder.

In the paper we obtain the differentiable solution of the problem. We interpolate the velocities and pressures at the points of the inner surface of the cylinder and find the velocities and pressures at the points of the flow.

In Section 2, the coordinates of the desired vector of velocity and the pressure are supposed to be polynomials with respect to the coordinate $\tilde{z}$ when the generatrix of the cylinder is parallel to the axis $O\tilde{Z}$. So the coefficients of the polynomials are the solutions of differential equations. These coefficients are found in Section 3 successively step by step beginning with the coefficients with the highest numbers. The following Section 4 contains the solution for the case when the velocities are given at three levels on the inner side of the circular cylinder with inner radius 1 and the axis of symmetry $O\tilde{Z}$. Section 5 reduces reconstruction of velocities and pressures for a viscous fluid from velocities given at a finite number of points on the inner surface to the method described in the previous sections.

This method is the analogue of the method, presented in [1,2], where the boundary value problems of the theory of elasticity have been solved.

2. Formulation of the problem and analysis

Let Ω be the inner surface of a hollow cylinder in the $XY\tilde{Z}$ space with the generatrix parallel to the axis $O\tilde{Z}$. The intersection of Ω with the plane XOY is the closed curve C which is the boundary of the finite domain D. Consider

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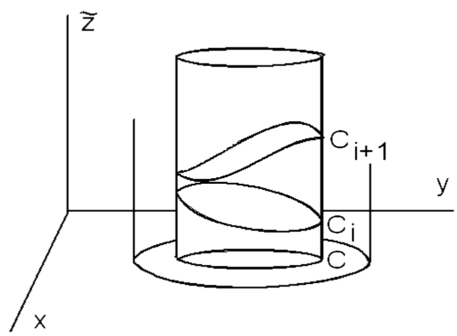
stable slow flow of the incompressible viscous fluid in the cylinder. Some interference on the inner sides of the cylinder influences the velocity of the fluid. Let C_j, $j = 0, \ldots, n$, be the closed curves on the surface Ω which have the one-to-one projection on C and do not intersect each other.



Given the velocities at the points of the curves C_j, $j = 0, 1, \ldots, n$, and the pressures at two points $(x_0, y_0, \tilde{z}_{j_k})$, $k = 1, 2$, of the curves C_{j_k}, $j_k = 0, 1, \ldots, n$, it should be possible to restore the velocities and pressures at any point of the flow.

We denote the components of the velocity by $u(x, y, \tilde{z})$, $v(x, y, \tilde{z})$, $w(x, y, \tilde{z})$ and the pressure by $p(x, y, \tilde{z})$. We suppose the Reynolds number of the flow to be low so that we can apply Stokes equations to describe this flow, together with the incompressibility equation:

$$\frac{\partial p}{\partial x} = \mu \Delta u, \quad \frac{\partial p}{\partial y} = \mu \Delta v, \quad \frac{\partial p}{\partial \tilde{z}} = \mu \Delta w,$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial \tilde{z}} = 0, \quad (x, y) \in D. \tag{1}$$

Let the parametric equation of the curves C_j, $j = 0, 1, \ldots, n$, be $\{x(s), y(s), \tilde{z}_j(s)\}$, the parameter $s \in [0, l]$ being the arc length of the curve C, $x_0 = x(0)$, $y_0 = y(0)$, $\tilde{z}_{j_k} = \tilde{z}_{j_k}(0)$, $k = 1, 2$.

We have the following given data:

$$u(x, y, \tilde{z})|_{(x,y) \in C_j} = \tilde{u}_j(s), \quad v(x, y, \tilde{z})|_{(x,y) \in C_j} = \tilde{v}_j(s),$$
$$w(x, y, \tilde{z})|_{(x,y) \in C_j} = \tilde{w}_j(s), \quad s \in [0, l], \tag{2}$$

$$p(x(0), y(0), \tilde{z}_{j_k}(0)) = \tilde{p}_{j_k}, \quad k = 1, 2. \tag{3}$$

We search for the solution in the form

$$u(x, y, \tilde{z}) = \sum_{k=0}^{n} u_k(x, y)\tilde{z}^k, \quad v(x, y, \tilde{z}) = \sum_{k=0}^{n} v_k(x, y)\tilde{z}^k,$$
$$w(x, y, \tilde{z}) = \sum_{k=0}^{n} w_k(x, y)\tilde{z}^k, \quad p(x, y, \tilde{z}) = \sum_{k=0}^{n} p_k(x, y)\tilde{z}^k. \tag{4}$$

We put the velocity in the form of Eqs. (4) into equalities (2) and obtain the systems with the Vandermonde determinant which lead to the values

$$u_k(x, y)|_{(x,y) \in C} = \hat{u}_k(s), \quad v_k(x, y)|_{(x,y) \in C} = \hat{v}_k(s),$$
$$w_k(x, y)|_{(x,y) \in C} = \hat{w}_k(s), \quad k = 0, 1, \ldots, n. \tag{5}$$

Now we put the velocity and the pressure in the form of Eqs. (4) into Eqs. (1) and compare the coefficients with the same powers of $\tilde{z}$ at the right and the left part of each equation. We have the following equations in the unknown coefficients:

$$\frac{\partial p_k}{\partial x} = \mu \Delta u_k + \mu(k+1)(k+2)u_{k+2},$$
$$\frac{\partial p_k}{\partial y} = \mu \Delta v_k + \mu(k+1)(k+2)v_{k+2},$$
$$(k+1)p_{k+1} = \mu \Delta w_k + \mu(k+1)(k+2)w_{k+2},$$
$$\frac{\partial u_k}{\partial x} + \frac{\partial v_k}{\partial y} + (k+1)w_{k+1} = 0 \tag{6}$$

for each $k = 0, 1, \ldots, n$, where $u_{n+2} \equiv u_{n+1} \equiv v_{n+2} \equiv v_{n+1} \equiv w_{n+2} \equiv w_{n+1} \equiv p_{n+1} \equiv 0$.

3. Finding the coefficients

We solve Eqs. (6) successively beginning with the highest number $k = n$.

3.1. Examination of coefficients of $\tilde{z}^n$

The third equation in (6) leads to the equality $\Delta w_n = 0$, $(x, y) \in D$, so we have the Dirichlet problem for the Laplace equation where the boundary condition is the third of Eqs. (5). The solution of this problem is $w_n(x, y) = \Re\, r_n(z)$, $z = x + \mathrm{i}y$, where $r(z)$ is the analytic function in D. The first two of Eqs. (6) with the help of the fourth equation lead to the system

$$\begin{cases} \dfrac{\partial p_n}{\partial x} + \dfrac{\partial}{\partial y}\left[\mu\left(\dfrac{\partial v_n}{\partial x} - \dfrac{\partial u_n}{\partial y}\right)\right] = 0, \\ \dfrac{\partial p_n}{\partial y} - \dfrac{\partial}{\partial x}\left[\mu\left(\dfrac{\partial v_n}{\partial x} - \dfrac{\partial u_n}{\partial y}\right)\right] = 0, \end{cases}$$

which is equivalent to the equation

$$\frac{\partial}{\partial \bar{z}}\left[p_n + \mathrm{i}\mu\left(\frac{\partial u_n}{\partial y} - \frac{\partial v_n}{\partial x}\right)\right] = 0.$$

Therefore,

$$p_n(x, y) = \Re\, h_n(z), \quad \frac{\partial u_n}{\partial y} - \frac{\partial v_n}{\partial x} = \frac{1}{\mu}\Im\, h_n(z), \tag{7}$$

where $h_n(z)$ is analytic in D.

Now if we note that

$$\frac{\partial}{\partial z}(u_n + \mathrm{i}v_n) \equiv \frac{1}{2}\left(\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial v_n}{\partial x} - \frac{\partial u_n}{\partial y}\right),$$

we obtain from the fourth equation of (6) and from the second equation of (7) the following relation:

$$\frac{\partial}{\partial z}(u_n + \mathrm{i}v_n) = -\frac{1}{4\mu}[h_n(z) - \overline{h_n(z)}].$$

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