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Dynamic motion of a conical frustum over a rough horizontal plane

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ABSTRACT

An analytical and numerical study of the dynamic motion of a conical frustum over a planar surface is presented resulting to a non-linear system of ordinary differential equations. Wobbling and rocking components of motion are discussed in detail concluding that, in general, the former component dominates the latter. For small inclination angles an asymptotic approximation of the angular velocities is possible, revealing the main characteristics of wobbling motion and its differences from rocking. Connection is made of the analysis with the behavior of the ancient classical columns, whose three dimensional dynamic response challenges the accuracy of the two dimensional models, usually applied in practice. The consideration of such discrete-blocky systems can benefit from the present study, through qualitative results and benchmarks for more complicated numerical methods, like the Distinct Element Method.

1. Introduction

The non-holonomic problem of a symmetric body by revolution, rolling on a planar surface, was first formulated by Routh in 1868 [1]. Since then, a significant number of papers appeared on this subject, focusing mostly on the study of the motion of a thin disk on a horizontal plane. The elaboration of the problem of the thin disk is presented in most classical textbooks of Dynamics [2–5] providing to the readers a typical example of non-holonomic motion. Noticing the early works of Appell [6] in 1900 (cf. also Korteweg [7]) and Gallop [8] in 1904, where analytic solutions are given in terms of Gauss hypergeometric and Legendre functions, we pass to the corps of papers of the current decade. The papers of O'Reilly [9], Kuleshov [10], Paris and Zhang [11], Kessler and O'Reilly [12], Borisov et al. [13], Le Saux et al. [14] provide a deep insight to the dynamic behavior of the thin disk. Equally important for the present study are also the papers of Koh and Mustafa [15] and Batista [16], which discuss the motion of a disk of finite thickness on a planar surface. In the latter papers the equations of motion of a cylindrical drum are derived and numerical simulations are performed.

In the present paper we deal with the case of a conical frustum, rolling on a rough horizontal surface. Using for the description of motion the Lagrange formulation, we distinguish between the wobbling and the rocking of the frustum and comment extensively on these components of motion. Stability analysis reveals the pure three dimensional character of the motion, while further approximations of the angular velocities under small inclination angles are elaborated to examine the main characteristics of the motion of the frustum. Finally, an attempt is made to interpret the dynamical behavior of ancient classical columns considering them as conical frustums with slightly different radii.

2. Equations of motion of a conical frustum on a rough horizontal plane

The formulation of the problem is based on the following assumptions:

a. The body is a homogeneous, rigid conical frustum.
b. The contact with the horizontal plane is assumed punctual. Notice that Kessler and O'Reilly [12] introduced a contact moment for simulating a 'flat' contact. This additional complication is not considered here, because rolling friction is disregarded.
c. At any given time the body is in contact with its horizontal planar base and only smooth transitions in time are considered.

2.1. Formulation of the system

The position of the body in the inertial frame $O(XYZ)$ is determined by the coordinates of the contact point $P(X_P, Y_P)$ and by the Euler angles (φ, θ, ψ), where φ is the precession angle, θ the inclination (nutation) angle and ψ the rotation about ζ-axis (Fig. 2). For $\theta=0$ the frustum comes into contact with the horizontal plane by whole base. Hence, the motion is restricted in the interval $\theta \in [0,\pi/2]$.

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If the frustum rolls without sliding then the velocity of the contact point $P(X_P, Y_P)$ is:

$$V_P = -R\dot{\psi} \tag{1}$$

where R is the radius of the base of the drum. Applying the Frobenius criterion, it may be easily proven that constraint (1) is non-holonomic.

Ground accelerations can also be considered by introducing the additional inertia terms

$$\dot{X}_P + R\dot{\psi}\cos\varphi = \alpha_X^{gr}\dot{u}_{gr}(t), \quad \dot{Y}_P + R\dot{\psi}\sin\varphi = \alpha_Y^{gr}\dot{u}_{gr}(t) \tag{2}$$

where α_X^{gr}, α_Y^{gr} are two scalar quantities, constant in time, that express the direction of the ground acceleration $\ddot{u}_{gr}(t)$.

Given the frictional law of the materials in contact (eg. Coulomb friction), the estimation of the sliding velocity is feasible by combining the velocity of the point P, regarded as a point of the frustum, with the frictional forces developed at the contact. However, this formulation extends the limits and the scope of the present paper and it will not be pursued further hereafter. Numerical and parametric studies that include sliding are, of course, important for practical applications, as they supply quantitative information to be used for design purposes, but add little to the qualitative understanding of the basic dynamics of the system.

The angular velocity components of the body relative to $C(\xi\ \eta\ \zeta)$ are

$$\begin{aligned} \omega_\xi &= \dot{\theta} \\ \omega_\eta &= \dot{\varphi}\sin\theta \\ \omega_\zeta &= \dot{\varphi}\cos\theta + \dot{\psi} \end{aligned} \tag{3}$$

whilst the components relative to the central principal axes system $C(\bar{x}\ \bar{y}\ \bar{z})$ are

$$\begin{aligned} \omega_{\bar{x}} &= \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi \\ \omega_{\bar{y}} &= -\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\cos\psi \\ \omega_{\bar{z}} &= \dot{\varphi}\cos\theta + \dot{\psi} \end{aligned} \tag{4}$$

Notice the coincidence of $\omega_{\bar{z}}$ and ω_ζ, because $\bar{z} = \zeta$.

The coordinates of the center mass of the conical frustum in $O(X\ Y\ Z)$ are given in terms of the contact point coordinates $P(X_P, Y_P)$, by the following relations:

$$\begin{aligned} X_c &= X_P - \ell\cos(a+\theta)\sin\varphi \\ Y_c &= Y_P + \ell\cos(a+\theta)\cos\varphi \\ Z_c &= \ell\sin(a+\theta) \end{aligned} \tag{5}$$

where $\ell = \sqrt{R^2 + (1/4)k_1^2h^2}$, $\tan a = (k_1h/2R)$, $k_1 = 2(\bar{z}_{cm}/h)$, $\bar{z}_{cm} = ((3\beta^2+2\beta+1)/(\beta^2+\beta+1))(h/4)$, $\beta = r/R$, r and R are, respectively, the radii of the upper and lower rim of the conical frustum and h is its height (Fig. 1).

The velocity $\mathbf{V}_C$ of the center of the mass of the frustum yields

$$\begin{aligned} \dot{X}_C &= \dot{X}_P + \ell\dot{\theta}\sin(a+\theta)\sin\varphi - \ell\dot{\varphi}\cos(a+\theta)\cos\varphi \\ \dot{Y}_C &= \dot{Y}_P - \ell\dot{\theta}\sin(a+\theta)\cos\varphi - \ell\dot{\varphi}\cos(a+\theta)\sin\varphi \\ \dot{Z}_C &= \ell\dot{\theta}\cos(a+\theta) \end{aligned} \tag{6}$$

2.2. Dynamic equations of motion

The kinetic and the potential energy of the drum are

$$\begin{aligned} T &= \frac{1}{2}m\mathbf{V}_C^2 + \frac{1}{2}\boldsymbol{\omega}^{\mathbf{T}}\mathbf{I}_C\boldsymbol{\omega} \\ V &= mgz_C \end{aligned} \tag{7}$$

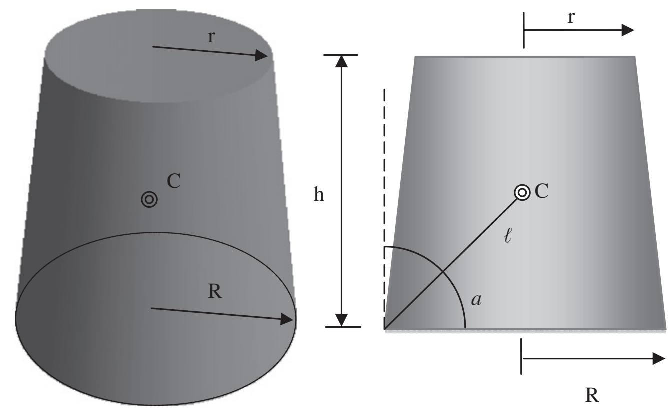

Fig. 1. The conical frustum: 3D and 2D view.

where $\mathbf{I}_C$ is the inertia tensor relative to $C(\bar{x}\ \bar{y}\ \bar{z})$

$$\mathbf{I}_C = \mathbf{I}_P - \begin{pmatrix} \frac{1}{4}k_1mh^2 + mR^2 & 0 & 0 \\ 0 & \frac{1}{4}k_1mh^2 + mR^2 & 0 \\ 0 & 0 & mR^2 \end{pmatrix},$$

$$\mathbf{I}_P = \begin{pmatrix} I_{\bar{x}}^P & 0 & 0 \\ 0 & I_{\bar{y}}^P & 0 \\ 0 & 0 & I_{\bar{z}}^P \end{pmatrix}$$

$$I_{\bar{z}}^P = \frac{1}{2}(k_3+2)mR^2, \quad I_{\bar{x}}^P = I_{\bar{y}}^P = \frac{1}{4}(k_3+4)mR^2 + \frac{1}{12}(k_2+3k_1^2)mh^2$$

$$k_2 = \frac{9}{20}\frac{\beta^4+4\beta^3+10\beta^2+4\beta+1}{\left(\beta^2+\beta+1\right)^2} \quad \text{and} \quad k_3 = \frac{3}{5}\frac{\beta^4+\beta^3+\beta^2+\beta+1}{\beta^2+\beta+1}$$

The inertia tensor $\mathbf{I}_P$ expresses the inertia moments of the body at the contact point P. For cylindrical drums it holds $k_1 = k_2 = k_3 = \beta = 1$.

Introducing the generalized coordinates $q_1 = \varphi$, $q_2 = \theta$, $q_3 = \psi$, $q_4 = X_P$ and $q_5 = Y_P$ the general form of the Lagrange equations for non-holonomic systems are

$$\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial\dot{q}_i}\right] - \frac{\partial(T-V)}{\partial q_i} - \sum_{j=1}^{2}\lambda_j B_{ji} = 0 \tag{8}$$

With λ_i we denote the Lagrange multipliers, while

$$B_{ji} = \frac{\partial\{\text{non-holonomic constraint equation}'j'\}}{\partial\dot{q}_i}, \text{ resulting to :}$$

$$\{B_{ji}\} = \begin{pmatrix} 1 & 0 & 0 & 0 & R\cos\phi \\ 0 & 1 & 0 & 0 & R\sin\phi \end{pmatrix}.$$

For convenience we introduce the following dimensionless quantities:

$$\begin{aligned} &\hat{h} = \frac{h}{R}, \quad \tau = t\sqrt{\frac{g}{R}}, \quad \hat{X} = \frac{X_P}{R}, \quad \hat{Y} = \frac{Y_P}{R}, \quad \hat{u}_{gr} = \frac{u_{gr}}{R}, \quad (.)' \equiv \frac{d(.)}{d\tau}, \\ &\hat{I}_k = \frac{I_k}{mR^2}, \quad \hat{\omega}_k = \omega_k\sqrt{\frac{R}{g}}, \quad \hat{T} = \frac{T}{mgR}, \quad \hat{V} = \frac{V}{mgR}, \quad \hat{E} = \frac{E}{mgR} \end{aligned} \tag{9}$$

where g is the acceleration of gravity and E the total energy of the system.

According to Eq. (8), the equations of motion are written in matrix notation

$$\mathbf{A}\cdot\mathbf{U} = \mathbf{B} \tag{10}$$