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Characterization of bifurcating non-linear normal modes in piecewise linear mechanical systems

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The non-linear modal properties of a vibrating 2-DOF system with non-smooth (piecewise linear) characteristics are investigated; this oscillator can suitably model beams with a breathing crack or systems colliding with an elastic obstacle. The system having two discontinuity boundaries is non-linearizable and exhibits the peculiar feature of a number of non-linear normal modes (NNMs) that are greater than the degrees of freedom. Since the non-linearities are concentrated at the origin, its non-linear frequencies are independent of the energy level and uniquely depend on the damage parameter. An analysis of the NNMs has been performed for a wide range of damage parameter by employing numerical procedures and Poincaré maps. The influence of damage on the non-linear frequencies has been investigated and bifurcations characterized by the onset of superabundant modes in internal resonance, with a significantly different shape than that of modes on fundamental branch, have been revealed.

1. Introduction

The classical modal analysis in linear dynamics can be extended to non-linear systems governed by smooth equations by introducing the concept of non-linear normal modes (NNMs). According to the classical definition given by Rosenberg [1,2], a NNM of an undamped system is defined as a synchronous periodic oscillation where all generalized coordinates of the system reach their extreme values or pass through the zeros simultaneously. The NNMs of a system are important because, in analogy to linear theory, resonance in forced systems typically occurs in the neighborhood of NNM frequencies. Hence, knowledge of the normal modes of a non-linear system can provide valuable insight regarding the position of its resonances, a feature of considerable engineering importance. Moreover, since the number of normal modes of a non-linear system may exceed its degrees of freedom (*superabundant NNMs*), certain forced resonances are essentially non-linear and have no analogies in linear theory; in such cases a linearization of the system might not be possible, or might not provide all the possible resonances that can be experienced.

In recent years, this item has revealed its importance also for piecewise-smooth mechanical systems (PSS): these systems are governed by sets of ordinary differential equations, which are smooth in regions of phase space; smoothness is lost as trajectories cross the boundaries between adjacent regions (*discontinuity boundaries*). Much effort in science and engineering has focused on these kinds of problems [3]. Typical mechanical applications include: oscillators colliding with a deformable or rigid stop [4,5], block assemblies connected by no-tension springs [6], beams with breathing cracks [7–10] and stick-slip mechanical systems with friction [11,12].

Particular attention has been devoted to the investigation of the non-linear modal properties of these systems. To this end, the classical Rosenberg's definition has been extended, to make it suitable for non-conservative and non-smooth systems: NNM is then defined as any periodic motion in which all generalized coordinates vibrate without necessarily passing through the zeros simultaneously [9,13]. Important contributions to the understanding of NNMs from a general point of view can be found in [8,14–21].

In a recent study [13], a discrete model of a beam with a breathing crack, an example of a non-linearizable system, has been dealt with using the asymptotic method of Lindstedt–Poincaré and limiting the analysis to the fundamental branch solutions and their stability. In [22], a similar system was then numerically investigated by means of the Poincaré map. Particular attention was given to the onset of superabundant modes along the backbones of the two fundamental modes.

The present paper aims to give a general picture of the basic non-linear dynamics of these systems. In particular the onset of superabundant modes at internal resonances will be characterized and it will be shown that the specific features of each

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bifurcation scenario peculiarly depend on the kind of the related internal resonance. The non-linear modal characteristics of a general 2-DOF piecewise-smooth mechanical system with two damage parameters are analyzed; the system can model the dynamics of an asymmetrically multi-cracked cantilever beam vibrating in bending and hence exhibiting a bilinear stiffness. The phase space of this dynamical system is divided into four regions by two discontinuity boundaries; in each region the system has a different smooth functional form of the vector field. Since the vector field is the same in the adjacent regions, whereas its Jacobian changes due to the bilinear stiffness, the problem belongs to a *continuous* PSS, according to the definition widely used in the literature [3]. The system at hand encompasses, as particular cases, the oscillators with a single damage parameter investigated in [9,13,22].

2. Mechanical system

2.1. Model description

The investigated system, Fig. 1, consists of a 2-DOF oscillator: two masses, m_1 and m_2, are connected by two piecewise linear springs of undamaged stiffness k_1 and k_2, and reduced stiffness $(1-\varepsilon_1)k_1$ and $(1-\varepsilon_2)k_2$, $(0\leq\varepsilon_i<1)$: the relevant restoring forces exhibit the bilinear behavior shown in Fig. 1b.

Assuming the displacements x_1 and x_2 as Lagrangian coordinates, the stiffness of the non-linear springs can be represented, Fig. 1b:

$$k_{bil,i}=k_i(1-H(\eta_i)\varepsilon_i),\ \ H(\eta_i)=\begin{cases}1 & \eta_i\geq 0\\ 0 & \eta_i<0\end{cases}\quad i=1,2 \tag{1}$$

where ε_1 and ε_2 are the damage parameters, H is the Heaviside function, and $\eta_1=x_1$ and $\eta_2=x_2-x_1$. Therefore, Fig. 1c, in the configuration plane (x_1, x_2) four regions with different stiffness properties are delimited by the following two boundaries:

$$\Sigma_1 :=\{\mathbf{x}\in \mathrm{R}^2 : \eta_1(\mathbf{x})=0\},\quad \Sigma_2 :=\{\mathbf{x}\in \mathrm{R}^2 : \eta_2(\mathbf{x})=0\} \tag{2}$$

As shown in Fig. 1c, in regions I and III the system exhibits only one spring with reduced stiffness at a time, whereas in regions II and IV the springs are both damaged or both undamaged.

2.2. Equations of motion

With reference to Fig. 1, the following equations of motion in time domain are found:

$$\begin{cases}m_1\ddot{x}_1+[k_{bil,1}+k_{bil,2}]x_1-k_{bil,2}x_2=0\\ m_2\ddot{x}_2+k_{bil,2}x_2-k_{bil,2}x_1=0\end{cases} \tag{3}$$

Introducing a state vector $y\equiv(y_1,y_2,y_3,y_4)$ and gathering the state variables of displacement and velocity of each mass $(y_1=x_1,\ y_2=\dot{x}_1, y_3=x_2,\ y_4=\dot{x}_2)$, Eqs. (3) can be written as

$$\begin{cases}\dot{y}_1=y_2\\ \dot{y}_2=(-(k_{bil,1}+k_{bil,2})y_1+k_{bil,2}y_3)/m_1\\ \dot{y}_3=y_4\\ \dot{y}_4=(k_{bil,2}y_1-k_{bil,2}y_3)/m_2\end{cases} \tag{4}$$

In the following, mass and stiffness ratios will be denoted:

$$\frac{m_2}{m_1}=\alpha,\quad \frac{k_2}{k_1}=\beta \tag{5}$$

The NNMs exhibited by the system are analyzed in the configuration space for different values of α and β and by means of procedures based on continuation techniques and Poincaré maps. The relevant bifurcations are studied as a function of the damage parameters ε_1 and ε_2.

3. Normal modes

3.1. Linear normal modes (LNM)

For $\varepsilon_1=\varepsilon_2=0$ the system is linear and exhibits the two LNMs $\mathbf{u_{01}}$ and $\mathbf{u_{02}}$: the relevant modal curves are straight lines passing through the origin, and their frequencies are

$$\omega^2_{01,02}=\frac{(\alpha+\beta+\alpha\beta)\mp\sqrt{(\alpha+\beta+\alpha\beta)^2-4\alpha\beta}}{2\alpha}\frac{k_1}{m_1} \tag{6}$$

The frequency ratio ω_{02}/ω_{01} for the undamaged oscillator will be denoted as r_0 and, according to Eq. (6), is given by

$$r_0=\frac{\omega_{02}}{\omega_{01}}=\sqrt{\frac{(\alpha+\beta+\alpha\beta)+\sqrt{(\alpha+\beta+\alpha\beta)^2-4\alpha\beta}}{2\sqrt{\alpha\beta}}} \tag{7}$$

The parameter r_0 uniquely depends on the nondimensional mass and stiffness parameters α and β of Eq. (5).

As it will be shown, the dynamic behavior exhibited by the system when $\varepsilon_1\neq 0$ and/or $\varepsilon_2\neq 0$ is strongly affected by r_0. In the subsequent analyses the following remarkable cases will be considered:

(i) $r_0=1.95$, below the (2:1) internal resonance;
(ii) $r_0=1/2(3+\sqrt{5})\cong 2.62$, below the (3:1) internal resonance. This value is exhibited by a shear-type frame with equal masses and equal stiffness;

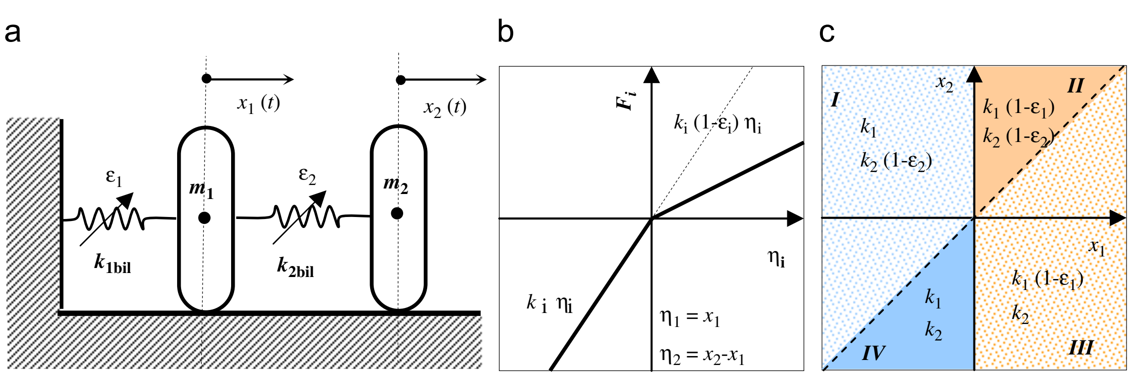

Fig. 1. (a) System model; (b) piecewise restoring forces ($\eta_1=x_1$; $\eta_2=x_2-x_1$); and (c) discontinuity boundaries in the physical plane (x_1, x_2).