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The bouncing motion of a superball between a horizontal floor and a vertical wall

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ABSTRACT

In earlier work [P.J. Aston, R. Shail, The dynamics of a bouncing superball with spin, Dyn. Sys. 22 (2007) 291–322] the problem of the possible back and forth motion of a superball thrown spinning onto a horizontal plane was considered in detail. In this paper the problem is extended to include a vertical wall. In particular motion of the superball where it bounces alternately on the floor and the wall several times is considered. Using the same physical model as in our previous work, a non-linear mapping is derived which relates the launch data of the $(n+1)$th floor bounce to that of the n th. This mapping is analysed both numerically and theoretically, and a detailed description is presented of various possible motions. Regions of initial conditions which result in a specified number of bounces against the wall are also considered.

1. Introduction

In a previous paper [1] two of the present authors considered in detail the mechanics of a superball bouncing back and forth on a rough horizontal plane. Reversals in direction in the horizontal motion of the ball result from the application of a tangential law of restitution at the point of impact of the ball and the plane. This concept was first introduced by Garwin [2] who used a tangential coefficient of restitution of one, which is not physically realistic. Garwin's model was modified by Cross [3] who employed a tangential coefficient of restitution α satisfying $0<\alpha<1$, with the horizontal velocity of the point of impact of the ball being reversed and reduced in magnitude by a factor of α in the impact. Further details of the physics of this model are given in [1], together with references to other theoretical and experimental work.

All who have experimented with a superball will have at sometime bounced the ball on the floor, followed by a bounce on a vertical wall. If the bounce on the wall occurs while the ball is still rising, it gives the ball some backspin, so that the direction of motion is reversed at the next bounce on the floor resulting in the ball hitting the wall a second time. With practice, the ball can be made to bounce between the floor and wall several times. Such motion is illustrated in the animations in Figs. 2, 3, 5, and 11. It is our purpose to give a theoretical investigation of such motions and the non-linear mappings which they engender. To this end we establish in Section 2 the basic equations governing the model. Essentially, each journey of the ball from floor to wall to floor, assumed to take place in the same vertical plane, comprises four events: (i) after launch from the floor the ball pursues a parabolic trajectory until it hits the wall, (ii) the rebound from the wall, (iii) the parabolic trajectory of the return journey to the floor and (iv) the impact with the floor which provides the launch data for the next excursion of the ball. The result of this analysis is the derivation of a non-linear mapping which relates the floor launch data (linear and angular velocity components of the ball and distance from the wall) to the same parameters after the next bounce on the floor.

In Section 3 some numerical trajectories of the non-linear mapping are computed and examples given of motions with various numbers of floor to wall bounces. Also illustrated are the parameter spaces of initial conditions required to produce various numbers of bounces off the wall. In Section 4 a scaling invariance is introduced which rewrites the non-linear map of Section 2 in terms of suitable canonical coordinates. This results in a three-dimensional non-linear map, a reduction in dimension by one from the original system.

Section 5 presents some numerical results for the regions of initial conditions which will result in a given number of bounces against the wall in the canonical variables, analogous to those of Section 3 for the original variables. The next two sections of the work analyse these numerical results in some detail, focussing on the behaviour of the mapping on two planes which comprise boundaries of the region of interest. The paper concludes by proposing a number of further questions related to the problem.

Before continuing to our analysis of the problem we have just described, we note that there are limitations to the model of the bounce of the superball that we use. It is recognised that the

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model introduced in [3] which was subsequently used in [1] and the present paper is an oversimplification of the physics of superball impact. Thus, the tangential coefficient of restitution α is known not to be constant, but to depend on factors such as the speeds, the angles of incidence and the angular velocities of the bodies in collision (see, for example, Cross [3,4], Labous et al. [5], Dong and Moys [6], Sondergaard et al. [7], although we note that the latter two studies were concerned with steel balls, not superballs). Further there is a number of competing models of the impact process which attempt to describe the slip and elastic restitution occurring over the area of contact of the impinging bodies. For example, Maw et al. [8] study in detail the elastic displacements of colliding spheres during impact, giving particular attention (via a classical elasticity mixed boundary-value problem) to the tangential tractions generated in the collision. Stronge et al. [9] model the collision by again considering the elastic impact region, which they represent by a deformable particle, the remainder of the system being treated as a rigid body. A very different approach to collision dynamics is that of Bibó et al. [10], who construct a mechanical model of a ball which can exhibit the back and forward bouncing studied in [1]. Basically they consider the ball to have a rigid core attached by torsion springs to an outer casing, each component being capable of rotation about a common axis. The outer layer mimics the surface layer of the ball whilst the inner part can store energy even if the outer layer is reduced to rest during the bounce. These and other models may be able to give a more realistic description of the bouncing process. However, despite the shortcomings of the Cross model of a bounce, it has the merit of enabling progress to be made in the analytical description of the title problem of this paper, and hence is to be preferred to other models which would lead to intractable mathematical and numerical situations.

2. The model equations

We consider the motion of a solid homogeneous superball of mass m and radius a, bouncing back and forth under gravity between a horizontal floor (f) and a vertical wall (w). The motion is assumed to be two-dimensional, and horizontal and vertical axes Ox and Oy are taken in the plane of motion of the centre, C, of the ball such that the horizontal floor is given by $y=-a$, $-a \le x < \infty$ and the vertical wall by $x=-a$, $-a \le y < \infty$. With this choice of coordinates, the ball centre C is restricted to the positive quadrant of the plane (see Fig. 1).

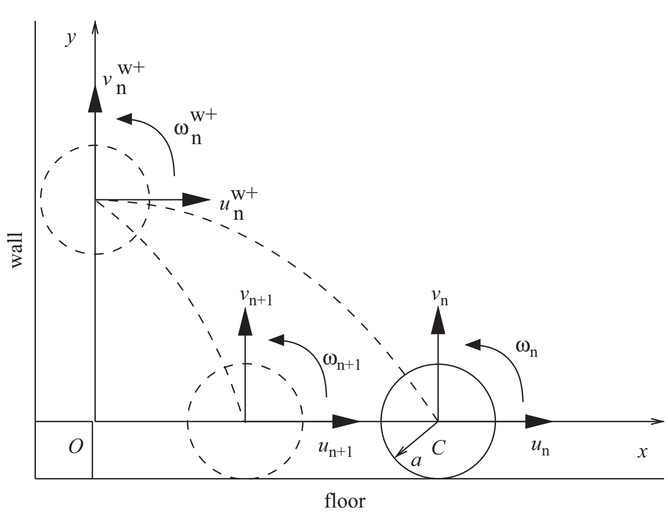

Fig. 1. The trajectory of the ball centre C and three successive impacts.

Since collisions occur at two separate surfaces it is essential to formulate a clear notation for describing the progress of the ball. Let u_n, v_n be the horizontal and vertical velocity components of the centre of the ball *immediately after* the nth bounce on the floor, and let ω_n, measured positive in the direction from Ox to Oy, be the angular velocity of the ball. Clearly $v_n > 0$ by definition and we require $u_n < 0$ in order for the ball to proceed *towards* the wall. Further, let $x_n + a$ denote the horizontal distance of the ball centre C from the wall at the nth bounce on the floor, and $y_n + a$ the height of C above the floor at the subsequent impact with the wall. After launch from the floor the centre of the ball describes a parabolic trajectory, and elementary mechanics shows that

$$y_n = -\frac{x_n}{u_n}\left(v_n + \frac{g x_n}{2u_n}\right). \tag{1}$$

Immediately prior to impacting the wall the linear and angular velocity components of the ball are denoted by u_n^{w-}, v_n^{w-} and ω_n^{w-}, and immediately after the collision with the wall these components are written u_n^{w+}, v_n^{w+} and ω_n^{w+}. The ball now returns to the floor, pursuing a parabolic trajectory, and reaches it with component velocities u_n^{f-}, v_n^{f-} and ω_n^{f-}, the centre C having travelled a horizontal distance x_{n+1}. Finally, the ball rebounds from the floor at the $(n+1)$th bounce with component velocities u_{n+1}, v_{n+1} and ω_{n+1}. Fig. 1 shows the trajectories of the centre C and the linear and angular velocities of the ball immediately after three successive impacts with the floor and wall structure.

During the flight of the ball between impacts any viscous or aerodynamic effects that might arise from the motion of the ball are assumed to be small and so are ignored; it follows that in any parabolic segment of the motion, the angular and horizontal velocities remain constant. In order to describe the interaction of the superball with the wall after the nth bounce on the floor, we introduce normal and tangential coefficients of restitution, e_w and α_w, both in the range (0,1), with a similar notation for the floor, the subscript f replacing w. e_w is the classical Newtonian coefficient whence, in the notation of the previous paragraph,

$$u_n^{w+} = -e_w u_n^{w-} = -e_w u_n. \tag{2}$$

In the direction tangential to the wall it is assumed, following Cross [3] and Aston and Shail [1], that the tangential velocity of the ball at the point of contact P_w with the wall is reversed and reduced in magnitude by a factor α_w. This condition gives

$$v_n^{w+} - a\omega_n^{w+} = -\alpha_w(v_n^{w-} - a\omega_n^{w-}) = -\alpha_w(v_n^{w-} - a\omega_n), \tag{3}$$

where

$$v_n^{w-} = v_n + \frac{g x_n}{u_n}. \tag{4}$$

A third model equation follows from the conservation of angular momentum in the bounce; taking moments about P_w, which obviates the need to introduce the impulsive friction and normal reaction at P_w, we have

$$\tfrac{2}{5} m a^2 \omega_n^{w+} + m a v_n^{w+} = \tfrac{2}{5} m a^2 \omega_n + m a v_n^{w-}. \tag{5}$$

Eqs. (3)–(5) give

$$v_n^{w+} = \frac{2}{7}(1+\alpha_w) a\omega_n + \frac{1}{7}(5 - 2\alpha_w)\left(v_n + \frac{g x_n}{u_n}\right), \tag{6}$$

$$\omega_n^{w+} = \frac{5}{7a}(1+\alpha_w)\left(v_n + \frac{g x_n}{u_n}\right) + \frac{1}{7}(2 - 5\alpha_w)\omega_n, \tag{7}$$

and (6) and (7), together with (2), furnish the launch velocities for the rebound from the wall.

We now consider the return of the ball to the floor and its rebound. The initial height of the centre of the ball above Ox is y_n, given by (1), and its horizontal range is x_{n+1}. Again, elementary