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On the effective stiffness of plates made of hyperelastic materials with initial stresses

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ABSTRACT

Within the framework of the direct approach to the plate theory we consider the infinitesimal deformations of a plate made of hyperelastic materials taking into account the non-homogeneously distributed initial stresses. Here we consider the plate as a material surface with 5 degrees of freedom (3 translations and 2 rotations). Starting from the equations of the non-linear elastic body and describing the small deformations superposed on the finite deformation we present the two-dimensional constitutive equations for a plate. The influence of initial stresses in the bulk material on the plate behavior is considered.

0. Introduction

Thin-walled engineering structures made of hyperelastic porous materials, such as polymer foams, have different applications in the last decades [1–4]. A polymer foam is a cellular structure consisting of a solid polymer, for example polyurethane, etc., containing a large volume fraction of gas-filled pores. There are two types of foams. One is the closed-cell foam, while the second one is the open-cell foam. The defining characteristics of foams is the very high porosity: typically well over 80%, 90% and even 98% of the volume consists of void spaces. The porosity and the topology of a foam determine the other properties, such as for example, Young's modulus, etc.

Polymer foams may demonstrate very large elastic strains. Hence such foams may be considered as a non-linear hyperelastic material. Different models allowing the description of large hyperelastic deformations of foams are proposed in the literature [3,4]. The existing models of foams may be classified as follows. The first type of models bases on the detailed considerations of the foam cell deformation taking into account the cell structure, the properties of cell walls and struts, the pressure change in the closed cells, etc., see [3–9] among others. The famous Kelvin model of foam belongs to this type. On the other hand the computational efforts may be significant and there is hard to establish experimentally the real material properties of cells. The second class of models use the description of a foam as the continuum media. Within the framework of this type models, one takes into account the structure of foam cells, the solid material and gas properties and other parameters in the constitutive equations at whole. Ogden's material model is applied for the finite deformations of hyperelastic foams, see [4,10–17]. Both types of models of hyperelastic foams have advantages and disadvantages. Further we apply the second approach using Ogden's material model of hyperelastic material for the moderate large strain and for the low level of stress field.

There are many plate-like engineering structures made of foams, for example sandwich plates with a core made of foam, laminates, etc., see [3,4] for details. Within the framework of the theory of plates and shells [18–23] the theory of elastic plates with non-homogeneous distribution of the porosity (functionally graded plates) is developed in [24] while the theory of viscoelastic plates is presented in [25,26], see also [27].

For the structures under consideration the initial stresses may influence on the plate behavior. The mechanics of the prestressed three-dimensional solids is developed in numerous papers and books, see [13,28–32] among others. The aim of this paper is to extend the results of [24] to plates made of material with internal stresses using the theory of small deformations superposed on finite deformation presented in the mentioned works. Let us note that the Kirchhoff–Love linear theory of shells made of prestressed material was earlier developed in a number of papers, see for example [30,33–36]. Here we consider the theory of plates taking into account the transverse shear deformations like in the theories proposed by Reissner [37,38] and Mindlin [39], see also the review [40].

The paper is organized as follows. In Section 1 we recall the basic equation of the three-dimensional theory of non-linear elasticity. We present here the governing equations describing the infinitesimal deformations of a prestressed body. Further in Section 2 we consider

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the linearized equilibrium equations of a three-dimensional plate-like body with the initial stresses depending on the thickness coordinate. In Section 3 we give the statement of the two-dimensional boundary-value problems for the linear plate theory.

1. Basic equations of 3D non-linear elasticity

Following [13,29–32] in this section we present the general equations governing small (incremental) deformations superimposed on a finite homogeneous deformation in an compressible elastic material. The Eulerian equilibrium equations of the non-linear body are given by the relations

$$\operatorname{div}\boldsymbol{\tau}+\rho\boldsymbol{f}=\mathbf{0},\quad \boldsymbol{\tau}=J^{-1}\mathbf{F}\cdot\mathbf{S},\quad \mathbf{S}=\frac{\partial W}{\partial \mathbf{F}}, \tag{1}$$

where div is the divergence operator in the actual configuration χ, $\boldsymbol{\tau}$ the Cauchy stress tensor, $\mathbf{S}$ the first Piola–Kirchhoff stress tensor, ρ the material density in the actual configuration, $\boldsymbol{f}$ the body force vector per unit mass, W the strain-energy function (per unit volume), $J=\det\mathbf{F}$, and $\mathbf{F}$ is the deformation gradient defined as in [13]. Note that here we use the notation $\mathbf{A}\cdot\boldsymbol{a}$ and $\mathbf{A}\cdot\mathbf{B}$ for the second-order tensors $\mathbf{A}$ and $\mathbf{B}$, and a vector $\boldsymbol{a}$ instead of the alternative way $\mathbf{A}\boldsymbol{a}$, and $\mathbf{AB}$, respectively. Further we assume the isotropic behavior of the material, so we use the constitutive equation in the following form:

$$W=W(I_1,I_2,I_3), \tag{2}$$

where I_1, I_2, I_3 are the principal invariants of the left Cauchy–Green deformation tensor $\mathbf{b}=\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}$ or the right Cauchy–Green deformation tensor $\mathbf{c}=\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$, defined by

$$I_1=\operatorname{tr}\mathbf{b}=\operatorname{tr}\mathbf{c}=\lambda_1^2+\lambda_2^2+\lambda_3^2,$$

$$I_2=\frac{1}{2}[\operatorname{tr}^2\mathbf{b}-\operatorname{tr}\mathbf{b}^2]\frac{1}{2}[\operatorname{tr}^2\mathbf{c}-\operatorname{tr}\mathbf{c}^2]=\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_3^2,$$

$$I_3=\det\mathbf{b}=\det\mathbf{c}=\lambda_1^2\lambda_2^2\lambda_3^2.$$

Here λ_1, λ_2, λ_3 are the principal stretches, tr denotes the trace of a second-order tensor, and $(\ldots)^{\mathrm{T}}$ denotes transposed. λ_1, λ_2, λ_3 may be also considered as the arguments of the strain function W:

$$W=W(\lambda_1,\lambda_2,\lambda_3).$$

For the isotropic material $\mathbf{S}$ and $\boldsymbol{\tau}$ are given by the relations

$$\mathbf{S}=2\frac{\partial W}{\partial \mathbf{c}}\cdot\mathbf{F}^{\mathrm{T}}=(f_0\mathbf{c}^{-1}+f_1\mathbf{I}+f_2\mathbf{c})\cdot\mathbf{F}^{\mathrm{T}},$$

$$\boldsymbol{\tau}=J^{-1}\mathbf{F}\cdot\mathbf{S}=f_0\mathbf{I}+f_1\mathbf{b}+f_2\mathbf{b}^2, \tag{3}$$

where $\mathbf{I}$ is the unit second-order tensor, f_0, f_1, f_2 are functions which may be expressed as combinations of the partial derivatives of W with respect to I_i or λ_i, see [13,29,30] for details.

For the description of the non-linear behavior of polymeric foams the following constitutive equation is widely used [4]

$$\begin{aligned}W&=\sum_{i=1}^{N}\frac{2\mu_i}{\alpha_i^2}\left[\operatorname{tr}\mathbf{b}^{\alpha_i/2}-3+\frac{1}{\beta_i}(\det\mathbf{F}^{-\alpha_i\beta_i}-1)\right]\\&=\sum_{i=1}^{N}\frac{2\mu_i}{\alpha_i^2}\left[\lambda_1^{\alpha_i}+\lambda_2^{\alpha_i}+\lambda_3^{\alpha_i}-3+\frac{1}{\beta_i}(J^{-\alpha_i\beta_i}-1)\right],\end{aligned} \tag{4}$$

where μ_i, α_i, β_i are the elastic moduli ($i=1,\ldots,N$). Here

$$\mu=\sum_{i=1}^{N}\mu_i.$$

μ denotes the initial shear modulus, while the initial bulk modulus k is given by

$$k=\sum_{i=1}^{N}2\mu_i\left(\beta_i+\frac{1}{3}\right).$$

The model (4) was originally proposed by Ogden [11,12], see also [4,16,10,17,15] among others, where Ogden's model is used. For some special choice of the values μ_i, α_i, β_i and N, Ogden's strain function W reduces to various others models applied in the non-linear elasticity (neo-Hookean, Varga, Mooney–Rivlin, Blatz–Ko constitutive equations, etc.).

Using the identity Div $(J^{-1}\mathbf{F})=\mathbf{0}$ we transform Eqs. (1) to the Lagrangian form

$$\operatorname{Div}\mathbf{S}+\rho_0\boldsymbol{f}=\mathbf{0}, \tag{5}$$

where Div is the divergence operator in the reference configuration and ρ_0 the density in this configuration.

Let us consider the equilibrium equations of a prestressed body. In other words, we introduce the small deformations superposed on the finite deformation. Let $\boldsymbol{x}$ be the known position vector in the actual configuration χ while $\boldsymbol{x}+\boldsymbol{w}$ is the position vector in another actual configuration $\chi^\star$ which differs from χ by the infinitesimal vector $\boldsymbol{w}$.

The linearization of Eq. (5) results in [13,29–32]

$$\operatorname{Div}\mathbf{S}^\star+\rho_0\boldsymbol{f}^\star=\mathbf{0}, \tag{6}$$

where

$$\mathbf{S}^\star=\frac{\partial^2 W}{\partial\mathbf{F}\partial\mathbf{F}}\cdot\cdot\,\mathbf{F}^{\star\mathrm{T}},\quad \mathbf{F}^\star=\operatorname{Grad}\boldsymbol{w},$$

and Grad is the gradient operators in the reference configuration, $\boldsymbol{f}^\star$ is the small additional body force acting in the actual configuration $\chi^\star$, and $\cdot\cdot$ is the double dot (inner) product.

The Lagrangian linearized equilibrium equation (6) may be transform to the Eulerian form

$$\operatorname{div}\boldsymbol{\Theta}+\rho\boldsymbol{f}^\star=\mathbf{0}, \tag{7}$$

where $\boldsymbol{\Theta}$ is the linearized stress tensor given by formulas [29,31]

$$\boldsymbol{\Theta}=J^{-1}\mathbf{F}\cdot\mathbf{S}^\star.$$

For example, let us consider the derivation procedure of $\mathbf{S}^\star$ and $\boldsymbol{\Theta}$ for the special case of (4) with $N=1$, $\alpha_1=2$, $\mu_1=\mu$, $\beta_1=\beta$. Here we have the constitutive relations

$$W=\frac{\mu}{2}\left[\operatorname{tr}\mathbf{c}-3+\frac{1}{\beta}(J^{-2\beta}-1)\right],$$

$$\mathbf{S}=\mu\mathbf{F}^{\mathrm{T}}-\mu J^{-2\beta}\mathbf{F}^{-1},\quad \boldsymbol{\tau}=\mu J^{-1}\mathbf{b}-\mu J^{-2\beta-1}\mathbf{I}. \tag{8}$$

Using the latter relations and the formula $J^\star=J\operatorname{div}\boldsymbol{w}$ we established the following relations for $\mathbf{S}^\star$ and $\boldsymbol{\Theta}$

$$\mathbf{S}^\star=\mu\mathbf{F}^{\star\mathrm{T}}+\mu J^{-2\beta}\mathbf{F}^{-1}\cdot\mathbf{F}^\star\cdot\mathbf{F}^{-1}+2\mu\beta J^{-2\beta}(\operatorname{div}\boldsymbol{w})\mathbf{F}^{-1},$$

$$\boldsymbol{\Theta}=\mu J^{-1}\mathbf{F}\cdot\mathbf{L}^{\mathrm{T}}\cdot\mathbf{F}^{\mathrm{T}}+\mu J^{-2\beta-1}\mathbf{L}+2\mu\beta J^{-2\beta-1}(\operatorname{div}\boldsymbol{w})\mathbf{I},$$

$$\mathbf{L}=\mathbf{F}^\star\cdot\mathbf{F}^{-1}\equiv\operatorname{grad}\boldsymbol{w}. \tag{9}$$

Here grad is the gradient operators in the actual (χ) configuration. Note that for the case $\mathbf{F}=\mathbf{I}$ Eqs. (9) reduce to Hooke's law

$$\mathbf{S}^\star=\boldsymbol{\Theta}=2\mu\boldsymbol{\varepsilon}+2\beta\mu\mathbf{I}\operatorname{tr}\boldsymbol{\varepsilon},\quad \boldsymbol{\varepsilon}=\frac{1}{2}(\mathbf{L}+\mathbf{L}^{\mathrm{T}}).$$

The equilibrium Eqs. (7) or (6) describe the prestressed solid deformable body as a result of infinitesimal deformations. From this point of view one may consider the relations $\boldsymbol{\Theta}=\boldsymbol{\Theta}(\mathbf{L})$ or $\mathbf{S}^\star=\mathbf{S}^\star(\mathbf{F}^\star)$ as the constitutive relations of the prestressed body. Of course, $\boldsymbol{\Theta}$ and $\mathbf{S}^\star$ depend also on the initial deformation gradient $\mathbf{F}$. Let us note that in the general case the tensors $\boldsymbol{\Theta}$ and $\mathbf{S}^\star$ are