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Generalisations of long wave theories for pre-stressed compressible elastic plates

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ABSTRACT

Generalisations of classical bending and extension are established for pre-stressed compressible elastic plates. In respect of the analogue of extension, the associated quasi-front is shown to be either advancing or receding, contrasting with the classical case. For the generalisation of bending, the long wave limit of the fundamental mode is non-zero; thus, unlike its classical counterpart, an associated quasi-front can, therefore, exist and is again noted to be either advancing or receding. In both cases appropriate leading order and higher order corrected governing equations are obtained. The ideas are illustrated through investigation of a model problem involving impact edge loading. For the generalised theory of bending, the leading order governing equation for the mid-surface deflection is used to establish the classical equation for wave propagation along an infinite string, with its second order refinement providing a second order correction. Motion within the vicinity of the thickness shear and thickness stretch resonance frequencies is also investigated. Special cases, in which either a stretch resonance and shear resonance frequency are very close, or the speeds of longitudinal and shear waves are very close, are also discussed.

1. Introduction

Lower dimensional plate theories have helped to elucidate qualitative features of static and dynamic structural response for many years. For the most part, certainly until relatively recently, these theories were only established within the framework of linear isotropic elasticity. The first attempts to extend these theories to include the influence of pre-stress were in [1], to which the reader is referred for some historical background. In [1] generalisations of classical bending and extension are established for a pre-stressed, incompressible elastic structures. Later, in [2], models for motion close to the cut-off frequencies were derived within the same constitutive framework. These models, to help, elucidate two-dimensional motion, were later extended to models for three-dimensional motion in incompressible pre-stressed layers, see [3,4] and also to problems involving slightly compressible elastic plates, see [5,6]. All of the above studies use the method of long wave asymptotic integration first developed in [7]. Our intention is to extend such studies and investigate the compressible pre-stressed problem. Within this context it is far easier to make direct comparison with the classical theories of bending and extension. Additionally, an interesting case arises in connection with motion near the shear and stretch resonance frequencies when the speeds of the shear and longitudinal waves are close. This is a phenomenon not possible within the classical linear isotropic context without strong convexity being violated, the bulk modulus being negative and Poisson's ratio tending to minus infinity in the limit of the speeds being equal.

This paper is organised as follows. In Section 2 the governing equations are reviewed and the dispersion relation associated with harmonic waves propagating in a layer with zero incremental traction on its faces is established. In Section 3, this relation is first very briefly investigated numerically and then long wave approximations are presented. In contrast to the classical linear isotropic case, the long wave limit of the anti-symmetric fundamental mode, the so-called long wave low-frequency limit, is non-zero. The implication is that an associated quasi-front exists. The long wave high-frequency region is also investigated, this being within the vicinity of the thickness shear and stretch resonance frequencies.

In Section 4, asymptotic integration is carried out in respect of low-frequency long wave motion, providing theories which are analogous to classical bending and extension. In the anti-symmetric motion case, the counterpart of classical bending, the leading order equation for the mid-plane deflection is shown to take the form of the classical wave equation. This second order equation is refined and an associated fourth order equation established. It is essential to use this higher order correction within the vicinity of the quasi-front. These ideas are illustrated through the setting up and solving of a model problem involving impact edge loading. In addition, it is demonstrated that if the normal pre-stress is zero, and the in-plane pre-stress a pure tension, the leading order equation then reduces to

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that of the classical equation for wave propagation along an infinite string. Within this context, the refined equation for the mid-surface deflection provides a higher order correction for the classical string equation. Asymptotic integration for symmetric motion is also carried out.

In Section 5, asymptotic integration is carried out for motion within the vicinity of either the thickness stretch or thickness shear resonance frequencies. In all cases, governing equations are derived for the long wave amplitudes. In Section 6, the first of two special cases is considered, namely the case in which one of the shear and one of the stretch resonance frequencies are very close. Modifications of the asymptotic integration procedure are made, with series expansions for the displacement components now in powers of the scaled wave number, rather than squares. The second special case, namely that for which the speeds of shear and longitudinal wave propagation coincide, is discussed in Section 7.

2. Governing equations and the dispersion relation

In this section we briefly review the appropriate governing equations and establish the dispersion relations; for further details the reader is referred to [8,9]. We consider a homogeneous, isotropic, compressible elastic layer of thickness $2h$ and infinite lateral extent. The layer possesses an initial unstressed configuration $\mathscr{B}_u$ and is subject to a homogeneous static deformation, resulting in the equilibrium pre-stressed state $\mathscr{B}_e$. A small time-dependent motion $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ is then superimposed on $\mathscr{B}_e$, resulting in the current configuration $\mathscr{B}_t$. A Cartesian coordinate system $Ox_1x_2x_3$, coincident with the principal axes of deformation in $\mathscr{B}_e$, is chosen, with Ox_2 normal to the layer's upper surface and origin O located in the mid-plane. The three principal stretches associated with the primary static deformation $\mathscr{B}_u \to \mathscr{B}_e$ are denoted by λ_1, λ_2 and λ_3. We utilise a plane-strain assumption, with $u_3 \equiv 0$ and u_1, u_2 independent of x_3. The governing equations of motion may be derived, see for example [9], in the form

$$\alpha_{11}u_{1,11} + \gamma_2 u_{1,22} + \beta u_{2,12} = \rho_e \ddot{u}_1,$$

$$\gamma_1 u_{2,11} + \alpha_{22}u_{2,22} + \beta u_{1,12} = \rho_e \ddot{u}_2, \tag{2.1}$$

within which

$$\alpha_{ij} = \mathscr{A}_{iijj}, \ \ i \in \{1,2\}, \quad \gamma_1 = \mathscr{A}_{1212}, \quad \gamma_2 = \mathscr{A}_{2121},$$

$$\beta = \alpha_{12} + \gamma_2 - \sigma_2 \tag{2.2}$$

with $\sigma_i, i \in \{1,2\}$ the principal Cauchy stresses in $\mathscr{B}_e$, ρ_e the material density in $\mathscr{B}_e$ and $\mathscr{A}_{ijkl}$ components of the fourth order elasticity tensor. A comma and a dot indicate differentiation with respect to x_1, x_2 and time t, respectively. Linearised measures of incremental traction, with outward unit normals along Ox_1 and Ox_2 in $\mathscr{B}_e$, have components

$$\tau_{1(1)} = \alpha_{11}u_{1,1} + \alpha_{12}u_{2,2}, \quad \tau_{2(1)} = (\gamma_2 - \sigma_2)u_{1,2} + \gamma_1 u_{2,1},$$
$$\tau_{1(2)} = \gamma_2 u_{1,2} + (\gamma_2 - \sigma_2)u_{2,1}, \quad \tau_{2(2)} = \alpha_{12}u_{1,1} + \alpha_{22}u_{2,2}. \tag{2.3}$$

Our initial concern is a layer with incrementally traction free upper and lower boundaries, indicating that $\tau_{1(2)}(\pm h) = \tau_{2(2)}(\pm h) = 0$. Solutions of the equations of motion are sought in the form of the travelling harmonic wave

$$(u_1, u_2) = (A, B)\mathrm{e}^{kqx_2}\mathrm{e}^{\mathrm{i}k(x_1 - vt)}, \tag{2.4}$$

where k is the wave number, v is the phase speed and q is to be determined.

Substituting the solutions (2.4) into the equations of motion (2.1), a system of linear homogeneous equations is obtained. This system possesses a non-trivial solution provided

$$\alpha_{22}\gamma_2 q^4 + \{\beta^2 - \alpha_{22}(\alpha_{11} - \bar{v}^2) - \gamma_2(\gamma_1 - \bar{v}^2)\}q^2$$
$$+ (\alpha_{11} - \bar{v}^2)(\gamma_1 - \bar{v}^2) = 0, \quad \bar{v}^2 = \rho_e v^2. \tag{2.5}$$

Solutions for the displacement components u_1 and u_2 may be represented as linear combinations of the four linearly independent functions $\exp(kq_ix_2)$ and $\exp(-kq_ix_2)$, $i \in \{1,2\}$, where $\pm q_1$, $\pm q_2$ are the four generally distinct and non-zero roots of (2.5). Substituting these solutions into the traction free boundary conditions, a system of four linear equations is obtainable. This system may be decomposed into two independent systems of two linear equations, corresponding to so-called anti-symmetric and symmetric motion. These two systems provide the two dispersion relations, which are expressible in the forms

$$q_1(\zeta_1 - \zeta_2 q_2^2)\tanh(q_1\eta) = q_2(\zeta_1 - \zeta_2 q_1^2)\tanh(q_2\eta) \tag{2.6}$$

and

$$q_1(\zeta_1 - \zeta_2 q_2^2)\tanh(q_2\eta) = q_2(\zeta_1 - \zeta_2 q_1^2)\tanh(q_1\eta), \tag{2.7}$$

respectively, where $\eta = kh$ and

$$\zeta_1 = (\alpha_{11} - \bar{v}^2)(\mathscr{E}_a - \bar{v}^2), \quad \zeta_2 = \alpha_{22}(\mathscr{E}_s - \bar{v}^2) \tag{2.8}$$

with

$$\mathscr{E}_a = \gamma_1 - \frac{(\gamma_2 - \sigma_2)^2}{\gamma_2}, \quad \mathscr{E}_s = \alpha_{11} - \frac{\alpha_{12}^2}{\alpha_{22}}. \tag{2.9}$$

In the case of anti-symmetric motion, u_1 and u_2 are expressible in terms of one arbitrary constant $\tilde{A}$, yielding

$$u_1 = \{H(q_2)\sinh(q_2\eta)\sinh(kq_1x_2) - H(q_1)\sinh(q_1\eta)\sinh(kq_2x_2)\}\tilde{A},$$

$$u_2 = \{F(q_1)H(q_2)\sinh(q_2\eta)\cosh(kq_1x_2)$$
$$- F(q_2)H(q_1)\sinh(q_1\eta)\cosh(kq_2x_2)\}\tilde{A} \tag{2.10}$$

with the exponential factor $\mathrm{e}^{\mathrm{i}k(x_1 - vt)}$ incorporated into $\tilde{A}$ and $F(q)$, $H(q)$ defined by

$$F(q) = \frac{\alpha_{11} - \bar{v}^2 - \gamma_2 q^2}{\beta \mathrm{i} q}, \quad H(q) = \beta(\alpha_{22}\mathrm{i}qF(q) - \alpha_{12}). \tag{2.11}$$

In the symmetric case, analogous solutions for u_1 and u_2 may be obtained by replacing sinh with cosh and cosh with sinh in (2.10). Finally in this section, necessary and sufficient conditions for strong ellipticity can be expressed in the form [9]

$$\alpha_{ii} > 0, \quad \gamma_i > 0, \quad (\alpha_{11}\alpha_{22})^{1/2} + (\gamma_1\gamma_2)^{1/2} \pm \beta > 0, \ \ i \in \{1,2\}. \tag{2.12}$$

3. Analysis of the dispersion relations

The dispersion relations (2.6) and (2.7) were first derived in [10], with a long wave asymptotic analysis later carried out in [11]. This section contains only the essential asymptotic results required in this paper. Our attention is focussed on long wave motion, implying that $\eta \to 0$. There are two types of asymptotic approximations needed to describe long wave motion, namely low- and high-frequency. The modes associated with these types of motion are usually referred to as fundamental modes and harmonics, respectively.

For numerical illustrations we make use of either a compressible neo-Hookean or Blatz-Ko material. The compressible neo-Hookean material has a strain energy function given by

$$W = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^3 - 2\ \ln(\lambda_1\lambda_2\lambda_3)) + \frac{\kappa}{2}(\lambda_1\lambda_2\lambda_3 - 1)^2, \tag{3.1}$$