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Buckling and post-buckling of extensible rods revisited: A multiple-scale solution

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ABSTRACT

An exact non-linear formulation of the equilibrium of elastic prismatic rods subjected to compression and planar bending is presented, electing as primary displacement variable the cross-section rotations and taking into account the axis extensibility. Such a formulation proves to be sufficiently general to encompass any boundary condition. The evaluation of critical loads for the five classical Euler buckling cases is pursued, allowing for the assessment of the axis extensibility effect. From the quantitative viewpoint, it is seen that such an influence is negligible for very slender bars, but it dramatically increases as the slenderness ratio decreases. From the qualitative viewpoint, its effect is that there are not infinite critical loads, as foreseen by the classical inextensible theory. The method of multiple (spatial) scales is used to survey the post-buckling regime for the five classical Euler buckling cases, with remarkable success, since very small deviations were observed with respect to results obtained via numerical integration of the exact equation of equilibrium, even when loads much higher than the critical ones were considered. Although known beforehand that such classical Euler buckling cases are imperfection insensitive, the effect of load offsets were also looked at, thus showing that the formulation is sufficiently general to accommodate this sort of analysis.

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1. Introduction

This paper should not begin without recalling the magisterial work by Euler [1] published 268 years ago, in which variational methods were applied to determine the "elastica" and buckling loads of inextensible rods, based on kinematical hypothesis suggested by Bernoulli [2]. Since then, the subject has been extensively studied, as seen in [3–8], due to its utmost relevance to the design of reticulated structures.

Also, this paper recasts and expands works written by the author more than 20 years ago [9,10]. The general non-linear equation of equilibrium of 2D Bernoulli–Euler elastic beam-columns subjected to end bending moment and compression force with possible loading offset (imperfection) is written in terms of cross-section rotations, taking into account axial stretching.

The general linearised equation is examined in order to evaluate critical loads for each one of the five classic Euler buckling cases, considering different constraint conditions [3,4]. Such critical loads are compared to the classical values for inextensible bars and conclusions are drawn with regard to the number of critical loads.

Next, the non-linear equation of equilibrium is recast and the post-buckling regime is surveyed using the method of multiple scales to produce a single explicit solution that is valid for any one of the five classic cases. Unlike the basic perturbation techniques used in non-linear statics, such as the straightforward expansion method (Poincaré's method) [11], which are barely capable of estimating the initial post-buckling response, the method of multiple scales [12] is able to supply a very accurate estimate of the displacements for loads much higher than the critical one, as seen when a comparison is made with results of numerical integration. The method of multiple scales is known for rendering uniformly convergent expansions, which is a most valuable feature in non-linear dynamics, where the independent variable (time) ranges from zero to infinity. In non-linear statics, the variation of the independent variable (co-ordinate along the bar axis) is comfortably limited to the bar length. It was already surprising to the author 20 years ago and so it is even more now, that very little attention has been given to the remarkable power of the multiple scales expansions to supply at the same time simple and accurate results in non-linear statics. In fact, only a few references on the use of the method of multiple scales in non-linear statics can be reported in the literature, as in [13,14].

Although the classical elastic Euler buckling cases are known to be imperfection insensitive [6,7], both the perfect and imperfect responses are inter-compared for the clamped-free and the hinged–hinged rods, as illustrative examples of the formulation generality.

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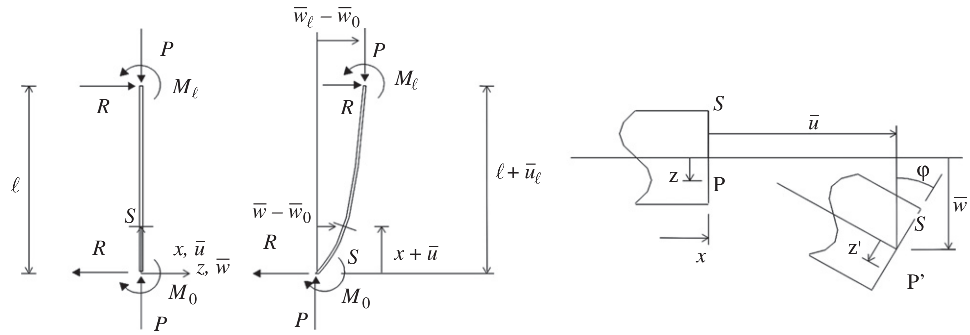

Fig. 1. (a) Prismatic elastic rod under bending and compression; (b) Bernoulli–Euler beam kinematics.

2. Non-linear equilibrium equation

The prismatic beam-column of Fig. 1(a), with length ℓ, cross-section area A and moment of inertia I, made of an elastic material of Young's modulus E, is considered. It is subjected to an initial axial compression P. It may be the case that end-bending moments come into play, as result of constraint conditions and/or load offsets. In the general case, to restore equilibrium with respect to moments, it may happen that end transversal forces R also appear.

Fig. 1(b) introduces the notation and refers to the Bernoulli–Euler kinematics, which is characterised by the following well-known relationships:

$$u = \bar{u} - z \sin \varphi,$$

$$w = \bar{w} - z(1 - \cos \varphi),$$

$$\sin \varphi = \frac{\bar{w}'}{\bar{\lambda}} \Rightarrow \bar{w}' = \bar{\lambda} \sin \varphi,$$

$$\cos \varphi = \frac{1 + \bar{u}'}{\bar{\lambda}} \Rightarrow 1 + \bar{u}' = \bar{\lambda} \cos \varphi, \tag{1}$$

where u and w stand for the axial and transversal displacements of a point P that in the undeformed configuration is given by (x,z); $\bar{u}$ and $\bar{w}$ are the corresponding displacements for the cross-section centroid at abscissa x; φ is the cross-section rotation at abscissa x; primes indicate derivation with respect to x. The axis stretching is given by

$$\bar{\lambda} = \sqrt{(1 + \bar{u}')^2 + (\bar{w}')^2}. \tag{2}$$

It can be shown [15]—for an elastic material obeying Hooke's law,[1] i.e., $\sigma = E(\lambda - 1)$, where λ is the stretching at the point (x,z)—that the normal force and the bending moment can be exactly evaluated as

$$N = EA(\bar{\lambda} - 1), \tag{3}$$

$$M = -EI\varphi'. \tag{4}$$

Considering the applied end loads, the normal force and the bending moment can also be written as

$$N = -P \cos \varphi + R \sin \varphi, \tag{5}$$

$$M = M_\ell - R[(\ell + \bar{u}_\ell) - (x + \bar{u})] - P(\bar{w}_\ell - \bar{w})$$
$$= M_0 + R(x + \bar{u}) + P\bar{w}, \tag{6}$$

[1] Filipisch and Rosales [16] consider other statements for Hooke's law, depending on which stress (engineering, second Piola–Kirchhoff, Cauchy) and strain (linear, Green, Almansi, Hencky) definitions are used.

where, without loss of generality, it was assumed in the last of (6) that $\bar{u}_0 = 0$ and $\bar{w}_0 = 0$. Hence, combining (3) and (5), as well as (4) and (6)

$$EA(\bar{\lambda} - 1) = -P \cos \varphi + R \sin \varphi, \tag{7}$$

$$-EI\varphi' = M_\ell - R[(\ell + \bar{u}_\ell) - (x + \bar{u})] - P(\bar{w}_\ell - \bar{w})$$
$$= M_0 + R(x + \bar{u}) + P\bar{w}. \tag{8}$$

After derivation with respect to x and taking (1) into account, (8) is rewritten as

$$-EI\varphi'' = R(1 + \bar{u}') + P\bar{w}' = \bar{\lambda}(R \cos \varphi + P \sin \varphi). \tag{9}$$

From (7), the axis stretching is

$$\bar{\lambda} = 1 - \left(\frac{P}{EA} \cos \varphi - \frac{R}{EA} \sin \varphi \right). \tag{10}$$

Finally, (10) in (9) leads to a second-order differential equation for the rotations:

$$EI\varphi'' + \left[1 - \left(\frac{P}{EA} \cos \varphi - \frac{R}{EA} \sin \varphi \right) \right] (R \cos \varphi + P \sin \varphi) = 0. \tag{11}$$

The corresponding non-dimensional equation is

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} + \frac{p}{\eta}[1 - p(\cos \varphi - \alpha \sin \varphi)](\alpha \cos \varphi + \sin \varphi) = 0, \tag{12}$$

$$p = \frac{P}{EA}, \quad \alpha = \frac{R}{P}, \quad \xi = \frac{x}{\ell}, \quad \eta = \frac{I}{A\ell^2}. \tag{13}$$

The exact non-linear Eq. (12) can be approximated up to the order ε^3, where $0 < \varepsilon \ll 1$, by

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} + \alpha_1\varphi + \varepsilon\alpha_2\varphi^2 + \alpha_3\varphi^3 = \varepsilon\alpha_0, \tag{14}$$

$$\varepsilon\alpha_0 = -\frac{\alpha p(1 - p)}{\eta}, \quad \varepsilon\alpha_2 = -\frac{\alpha p(1 - 4p)}{2\eta},$$

$$\alpha_1 = \frac{p(1 - p + \alpha^2 p)}{\eta}, \quad \alpha_3 = -\frac{p(1 - 4p + 4\alpha^2 p)}{6\eta}. \tag{15}$$

Notice that the non-homogeneous term and the coefficient of the quadratic term are scaled in (14) as of the order ε, since α is null or at least small compared to the unity in the five classic Euler buckling cases, which are the main concern of this study.

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