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An analytical study on the geometrical size effect on phase transitions in a slender compressible hyperelastic cylinder

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ABSTRACT

In this paper, we study phase transitions in a slender circular cylinder composed of a compressible hyperelastic material with a non-convex strain-energy function in a loading process. We aim to construct the asymptotic solutions based on an axisymmetrical three-dimensional setting and use the results to describe the key features observed in the experiments by others. By using a methodology involving coupled series-asymptotic expansions, we derive the normal form equation of the original complicated system of non-linear PDEs. Based on a phase-plane analysis, we manage to deduce the global bifurcation properties and to solve the boundary-value problem analytically. The explicit solutions (including post-bifurcation solutions) in terms of integrals are obtained. The engineering stress–strain curve plotted from the asymptotic solutions can capture the key features of the curve measured in some experiments. Our results can also describe the geometrical size effect as observed in experiments. It appears that the asymptotic solutions obtained shed certain light on the instability phenomena associated with phase transitions in a cylinder.

1. Introduction

Systematic experiments have been carried out on the uniaxial extensions of superelastic NiTi (a kind of shape memory alloy and also one kind of phase-transforming material) wires, strips and tubes [12,20,22,23,25,28]. It was found that the external force will induce phase transitions of the NiTi SMA wires, strips and tubes between austenite and martensite phases. During this phase transition process the new phase first nucleates at some special site of the specimen and then propagates gradually. The measured engineering stress–strain curves have some important key features: the nucleation stress occurs at a local maximum which is significantly larger than the Maxwell stress; following the nucleation stress there is a sharp stress drop; and afterwards there is a stress plateau. An important geometrical size effect was reported by Chang et al. [5] that the axial extent of the transformation front is of the order of the radius.

Theoretically, solid–solid phase transitions have been studied for a long time in the context of both continuum and lattice theories. As pointed out by Fu and Freidin [18], there are three major issues that need to be resolved: (i) characterization of materials that can support multiphase deformations; (ii) description of multiphase deformations under various loading conditions when the strain-energy function and the geometry of the elastic body are specified; and (iii) determination of the stability of multiphase deformations that can exist mathematically.

Many previous studies have been devoted to the resolutions of these issues. The seminal work of Ericksen [11], which considered a continuum one-dimensional stress problem, made clear that for a non-convex strain-energy function the solution with two phases can arise and there are multiple solutions. Based on the lattice model for a two-phase martensitic material, it is possible to deduce that in the related continuum model the strain-energy function has a double well; see [3]. In general, it is now understood that a necessary condition for phase transitions take place is that the material loses strong ellipticity at some deformation gradient (see also [1,2,14,19,21]). Still, to justify this point of view, it is desirable (probably necessary) to compare the analytical or numerical solutions based on this type of energy function with experimental results. For some specified strain-energy functions and geometry of the elastic bodies, detailed descriptions and stability analyses of multiphase deformations under various loading conditions have also been conducted by many previous works (see [10,13,15,16,18,26,29]).

In the papers of Dai and Cai [9] and Cai and Dai [4], phase transitions in a slender cylinder composed of a *special incompressible* elastic material were considered in a three-dimensional setting. A novel coupled series-asymptotic approach was utilized to reduce the field equations. A proper asymptotic model equation was derived,

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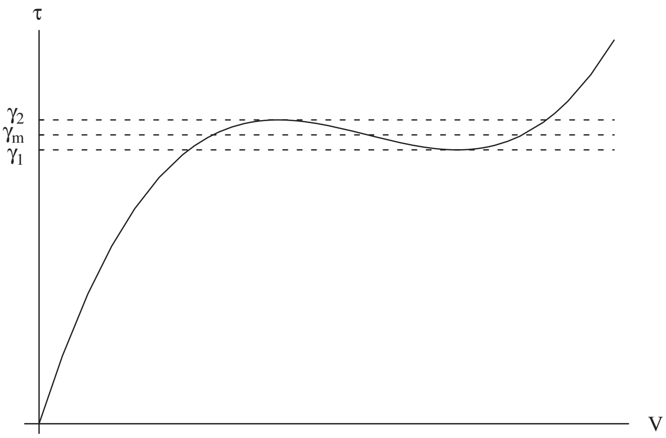

Fig. 1. The τ–V curve.

which took into account the influences of the radial deformation and traction-free boundary conditions. Analytical solutions for two boundary-value problems were obtained, and they could capture the key features observed in experiments.

In this paper, we study phase transitions in a slender cylinder composed of a *general compressible* elastic material due to extension. The emphasis is on describing the geometrical size effect through analytical (asymptotic) solutions. For that purpose, we consider the problem in a three-dimensional setting, different from the one-dimensional stress problem studied by Ericksen [11]. However, the strain-energy function is assumed to have the same property as that in [11], i.e., for a one-dimensional stress problem the stress–strain curve has a peak-valley combination (cf. Fig. 1). We aim at constructing the asymptotic solutions and using them to explain the experimental results.

Mathematically, to deduce the analytical solutions for the present problem, one needs to deal with coupled non-linear partial differential equations (PDEs) together with complicated boundary conditions. Further, the existence of multiple solutions (corresponding to instability phenomena (e.g., stress drop) observed in experiments) makes the problem even harder to solve. Here, by using the coupled series-asymptotic method (see also [7,8]), we derive the normal form equation (NFE) of the non-linear system of PDEs. Based on a phase-plane analysis, we construct the asymptotic solutions and extract from them important information on the deformed configurations, the nucleation stress, the instability phenomena and the transformation front. Comparisons with experimental results are made, which show that the asymptotic solutions can capture the key features of the experimental engineering stress–strain curves, the instability phenomena and the geometrical size effect as observed in experiments. The qualitative agreements give supporting evidence that the concave–convex nature of the engineering stress–strain curve indeed plays an important role in describing the solid–solid phase transitions in the present problem.

We point out that the model we proposed here is entirely macroscopic, it does not account for the influence of some important parameters such as grain size, alloy composition, etc. In our model, the effect of two phases coexist is smeared out so that the exact nature of the juxtaposition is lost. However, the analytical solutions we obtained here can also reflect the information of the phase states. In fact, the phase states can be represented by the axial strain values (cf. Fig. 9b): the low-strain region corresponds to the austenite phase, the high-strain region corresponds to martensite phase and the localization region where the axial strain varies at a high rate corresponds to the phase interface (where austenite and martensite coexist). Comparing with the discontinuous two-phase solutions obtained from some purely one-dimensional model (e.g., [11]), the continuous solutions we obtained here can provide some information on the phase transformation fronts. Determination of the stabilities for the solutions we obtained is still an open problem. Here, we just derive all the possible equilibrium solutions under certain restrictions. During the intervals where multiple solutions coexist, we shall use the minimum energy criterion to determine the preferred solution.

This paper is arranged as follows. In Section 2, we formulate the field equations by treating the slender cylinder as a three-dimensional object. In Section 3, we carry out a non-dimensionalization process to extract the important small variable and two small parameters which characterize this problem. Then we derive the NFE of the original governing non-linear PDEs in Section 4, through novel series and asymptotic expansions. In Section 5, we show that the Euler–Lagrange equation can also lead to the same NFE, which justifies our method of deriving this equation. In Section 6, we construct the asymptotic solutions for both a force-controlled and a displacement-controlled problem. The minimum energy criterion is used to determine the preferred solution. Based on the solutions obtained, we give some analysis on the geometrical size effect. Finally, some conclusions are drawn.

2. Three-dimensional field equations

We consider the axisymmetric deformations of a slender elastic cylinder subject to a static axial force at two ends. The lateral surface is traction-free and the end conditions will be considered later. The radius of the cylinder is a and the total length is l. We take the cylindrical polar coordinate system and denote (R,Θ,Z) and (r,θ,z) the coordinates of a material point of the cylinder in the reference and current configurations, respectively. The finite radial and axial displacements can be written as

$$U(R,Z)=r(R,Z)-R, \quad W(R,Z)=z(R,Z)-Z. \tag{2.1}$$

We introduce the orthonormal bases associated with the cylindrical coordinates and denote these by $\mathbf{E}_R$, $\mathbf{E}_\Theta$, $\mathbf{E}_Z$ and $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ in the reference and current configurations, respectively. Then the deformation gradient tensor $\mathbf{F}$ is given in these orthonormal bases by

$$\mathbf{F}=(1+U_R)\mathbf{e}_r\otimes\mathbf{E}_R+U_Z\mathbf{e}_r\otimes\mathbf{E}_Z+\left(1+\frac{U}{R}\right)\mathbf{e}_\theta\otimes\mathbf{E}_\Theta + W_R\mathbf{e}_z\otimes\mathbf{E}_R+(1+W_Z)\mathbf{e}_z\otimes\mathbf{E}_Z. \tag{2.2}$$

For an isotropic hyperelastic material, the strain-energy function Φ is a function of the three invariants I_1, I_2 and I_3 of the left Cauchy–Green strain tensor $\mathbf{B}=\mathbf{F}\mathbf{F}^{\mathrm{T}}$; that is, $\Phi=\Phi(I_1,I_2,I_3)$. We suppose that Φ is non-convex in a pure one-dimensional stress problem such that phase transition can take place. The nominal stress tensor $\mathbf{\Sigma}$ is given by

$$\mathbf{\Sigma}=\frac{\partial\Phi}{\partial\mathbf{F}}, \quad \Sigma_{ji}=\frac{\partial\Phi}{\partial F_{ij}}. \tag{2.3}$$

If the strains are small, it is possible to expand the nominal stress components in term of the strains up to any order. The formula containing terms up to the third-order material non-linearity is (cf. [17])

$$\Sigma_{ji}=a^1_{jilk}\eta_{kl}+\tfrac{1}{2}a^2_{jilknm}\eta_{kl}\eta_{mn}+\tfrac{1}{6}a^3_{jilknmqp}\eta_{kl}\eta_{mn}\eta_{pq}+O(|\eta_{st}|^4), \tag{2.4}$$

where η_{ij} are the components of the tensor $\mathbf{F}-\mathbf{I}$ and

$$a^1_{jilk}=\left.\frac{\partial^2\Phi}{\partial F_{ij}\partial F_{kl}}\right|_{\mathbf{F}=\mathbf{I}}, \quad a^1_{jilknm}=\left.\frac{\partial^3\Phi}{\partial F_{ij}\partial F_{kl}\partial F_{mn}}\right|_{\mathbf{F}=\mathbf{I}},$$

$$a^3_{jilknmqp}=\left.\frac{\partial^4\Phi}{\partial F_{ij}\partial F_{kl}\partial F_{mn}\partial F_{pq}}\right|_{\mathbf{F}=\mathbf{I}}$$