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Numerical approximation for a non-linear membrane problem

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ABSTRACT

We present a numerical study of large deformations of non-linearly elastic membranes. We consider the non-linear membrane model obtained by Le Dret and Raoult using Γ-convergence, in the case of a Saint Venant-Kirchhoff bulk material. We consider conforming P_1 and Q_1 finite element approximations of the membrane problem and use a non-linear conjugate gradient algorithm to minimize the discrete energy. We present numerical tests including membranes subjected to live pressure loads.

1. Introduction

The purpose of this article is to devise numerical approximations of large deformations of a non-linearly elastic membrane. The non-linear membrane model used here was obtained in [1], with refinements in [2]. The relevance of this model stems from the fact that it was derived from three-dimensional non-linear elasticity by means of a rigorous convergence method. Similar non-linear membrane models had already been obtained previously by Pipkin, directly in the context of standard two-dimensional membrane theory and using relaxation, see [3].

Our numerical study of the non-linear membrane model is made possible due to the explicit formula for the non-linear membrane energy given in [1] in the case of the Saint Venant-Kirchhoff bulk material. For a general bulk material, an explicit computation of the corresponding non-linear membrane energy entails the determination of the quasiconvex envelope of a function defined on the space of 3×2 matrices, a hopeless task as a general rule.

The work of [1] was motivated by [4], the first article to deal with a genuine dimension reduction in non-linear elasticity via a mathematical convergence result in the case of non-linearly elastic strings. It was followed in recent years by many, sometimes highly technical developments, including derivations of inextensional bending models and Von Kármán type models, see [5–7], among others. We are not however aware of the energies found in [1,2] ever being used in a numerical context, even though there are many numerical works on membranes and thin films, see [8,9] for numerical studies of Pipkin's model using a differential equation approach, including the case of a pressurized membrane [10], see also more or less ad hoc models designed for simplicity or efficiency, for instance in [11–13]. The modeling and numerical simulation of non-linear membranes is also attracting increasing interest for materials with more sophisticated material response than just non-linear elasticity, see [14] for finite strain, viscoelastic membranes.

One purpose of the present article is to advocate the use of a rigorously derived membrane energy to perform computations that are grounded on an indisputable three-dimensional model and are still efficient. Naturally, we then have to work with what is given by the asymptotic dimensional reduction procedure, and not with an ad hoc energy. This involves a little bit of mathematics, which needs to be carefully done, see Section 3 below.

This article is organized as follows: We first briefly present the results of [1,2]. We consider a three-dimensional hyperelastic homogeneous cylinder of thickness $2\varepsilon > 0$ made of a given Saint Venant-Kirchhoff material. The body is subjected to a dead loading body force density and a constant pressure differential on its upper and lower surfaces, and a boundary condition of place on its lateral surface. The three-dimensional non-linear elasticity equilibrium problem is formulated as a minimization problem for the total energy of the body.

Using Γ-convergence arguments, Le Dret and Raoult showed that deformations that almost minimize the three-dimensional total energy converge when the thickness ε of the body goes to zero towards deformations that minimize a non-linear membrane energy, see [1].

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The convergence takes place in a rescaled weak $W^{1,p}$ sense. The limit problem is two-dimensional, with values in $\mathbb{R}^3$.

The limit two-dimensional non-linear membrane energy is computed in two steps: First minimize the bulk stored energy function with respect to the third column vector of the deformation gradient—this step produces a function W_0 on the space of 3×2 matrices—then take the quasiconvex envelope of W_0. In the special case of a Saint Venant-Kirchhoff material, an explicit formula for this quasiconvex envelope QW_0 is available. This formula is expressed in terms of the right singular values of the membrane deformation. In [2], in addition to the case of curved membranes, the zero-thickness limit of a constant live pressure loading term is also computed.

In Section 3, we present a conforming finite element approximation of the membrane problem. We consider P_1 and Q_1 discretizations of the three Cartesian components of the deformation. In the P_1 case, deformations are approximated by piecewise affine, globally continuous functions on a triangulation of the domain. In the $Q1$ case, which is appropriate for rectangular membranes, a rectangular mesh is used with globally continuous approximations that are piecewise of partial degree less or equal to one on the rectangular elements. We prove the weak-$W^{1,4}$ convergence of the approximate solutions toward a solution of the continuous minimization problem.

The choice of available numerical methods to solve our FE problem is rather limited since the problem under study is highly non-linear and the membrane stored energy function is only of class C^1. Consequently, a method relying on second derivatives of the total energy such as the Newton–Raphson method cannot be appropriate. On the contrary, the non-linear conjugate gradient method with the Polak and Ribière variant seems to be well adapted to our problem. The convergence of the algorithm is guaranteed for a convex functional, which is only the case here when the pressure differential is zero. There is however a slight difficulty in computing the gradient of the stored energy function. We adapted Ball's results concerning the differentiability of frame-indifferent, isotropic functions on the space of $n \times n$ square matrices, see [15].

In Section 4, we present various numerical tests. Both P_1 and Q_1 elements are alternatively used. The first test is a circular membrane subjected to an upward pressure differential and clamped on its boundary. It should be noted that in our formulation, there is absolutely no need to track the deformed normal vector in order to take into account the live loading pressure differential. This is exemplified by the bubble-like deformation computed in this test which bulges out of the supporting circle. Note again that we have not seen our formulation of the pressure term used in a numerical context.

Next, we perform a few tests taken from [16]: a rectangular airbag inflated by an inner pressure and a square membrane attached by its four corners and subjected to a vertical point force applied at its center. As opposed to [16], our model cannot capture wrinkles in detail, because wrinkles are filtered out in the Γ-limit process, which in turn leads to a well-posed limit minimization problem. Such is the nature of weak convergence. However, wrinkled regions are captured. They are the membrane areas where the deformation gradient lies a region of 3×2 space where relaxation occurs, i.e., the quasiconvex envelope is such that $QW_0 < W_0$. This occurs in compression when at least one of the singular values is less than 1 and the other one is not too large, see [1] for details.

The last two tests are in the context of the modeling of fabrics. The first test is a square piece of fabric attached at its center and subjected to a vertical dead loading body force and the second is a tablecloth with no displacement allowed on the table surface. The obtained deformations develop folds and conical points where the normal vector is ill-defined, but this is of no consequence for our handling of the live pressure term.

Part of the results of this article concerning P_1 elements and without pressure differential were announced in [17].

2. The continuous problem

Let us briefly outline the results of [1,2], to which we refer the reader for more details. Let ω be an open, bounded subset of $\mathbb{R}^2$ with Lipschitz boundary. For all $\varepsilon > 0$, we consider a hyperelastic homogeneous body occupying the reference configuration $\Omega_\varepsilon = \omega \times]-\varepsilon, \varepsilon[$. We assume that the stored energy function of this body is a function $W: \mathbb{M}_3 \to \mathbb{R}$ which is continuous, coercive and satisfies growth conditions for an exponent $p \in]3, +\infty[$, where $\mathbb{M}_3$ is the space of real 3×3 matrices. We furthermore assume that the body is subjected to a dead loading body force density f and to a constant pressure differential $\varepsilon \Delta p$ on its upper and lower surfaces, which is a live load, that is to say a spatially constant pressure p_ε^+ on the upper surface and another spatially constant pressure p_ε^- on the lower surface such that $p_\varepsilon^+ - p_\varepsilon^- = \varepsilon \Delta p$. The equilibrium problem for this body may be formulated as a minimization problem for the energy

$$J_\varepsilon(\phi) = \int_{\Omega_\varepsilon} W(\nabla \phi)\, dx - \int_{\Omega_\varepsilon} f_\varepsilon \cdot \phi \, dx - P_\varepsilon(\phi), \tag{1}$$

where

$$P_\varepsilon(\phi) = \int_{\Omega_\varepsilon} \left[\pi_\varepsilon \det \nabla \phi + \frac{1}{3} \nabla \pi_\varepsilon \cdot (\operatorname{cof} \nabla \phi^{\mathrm{T}} \phi) \right] dx, \tag{2}$$

over a set of admissible deformations ϕ belonging to an appropriate Sobolev space and satisfying given boundary conditions of place on part of the lateral boundary. Here, π_ε is a C^1-function on $\bar{\Omega}_\varepsilon$ that takes the values $p_\varepsilon^\pm$ on the upper and lower surfaces. The term P_ε appearing in the energy accounts for the pressure load, see [18]. Note that this term incorporates the fact that a pressure load is a live load that follows the normal vector to the deformed body, without having to keep track of this normal vector. Dead loading tractions on the upper, lower and lateral surfaces can also easily be added as well as boundary conditions of place on part of the boundary.

In [1,2], see the latter for the pressure term, Le Dret and Raoult proved that a rescaled version of the above three-dimensional energy Γ-converges when the thickness 2ε of the membrane goes to zero in the sense of the weak topology of $W^{1,p}(\Omega; \mathbb{R}^3)$, thereby showing that minimizing deformations converge, in an appropriate sense, toward solutions of a two-dimensional minimization problem. The limit, two-dimensional non-linear membrane problem is described as follows.

Let $\mathbb{M}_{3,2}$ be the space of real 3×2 matrices. If $z_\alpha, \alpha = 1, 2$, are two vectors in $\mathbb{R}^3$, we note $(z_1|z_2)$ the matrix of $\mathbb{M}_{3,2}$ whose columns are the vectors z_α. For all $F = (z_1|z_2) \in \mathbb{M}_{3,2}$ and $z \in \mathbb{R}^3$, we note $(F|z)$ the matrix whose first two columns are z_α, and third column is z and write $(z|F)$ with a similar convention. We now define a function $W_0: \mathbb{M}_{3,2} \to \mathbb{R}$ by

$$W_0(F) = \inf_{z \in \mathbb{R}^3} W((F|z)). \tag{3}$$

The function W_0 is continuous and coercive. Let QW_0 be its quasiconvex envelope, see [19]. We introduce the space of admissible membrane displacements

$$\Phi_M = \{\psi \in W^{1,p}(\omega; \mathbb{R}^3); \psi(x_1, x_2) = (x_1, x_2, 0)^{\mathrm{T}} \text{ on } \partial\omega\}, \tag{4}$$

(this is for the case of a boundary condition of place on the whole lateral surface $\partial\omega \times]-\varepsilon, \varepsilon[$ in the three-dimensional problem, other conditions are enforced accordingly).

The limit non-linear membrane energy is then defined by

$$J(\psi) = 2 \int_\omega QW_0(\nabla \psi)\, dx_1\, dx_2 - \int_\omega f \cdot \psi \, dx_1\, dx_2 - \frac{\Delta p}{3} \int_\omega \det(\partial_1 \psi | \partial_2 \psi | \psi)\, dx_1\, dx_2, \tag{5}$$