International Journal of Non-Linear Mechanics 43 (2008) 925 – 932

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

An asymptotically stable collision-avoidance system

Jito Vanualailai[a,*], Bibhya Sharma[a], Shin-ichi Nakagiri[b]

[a]School of Computing, Information and Mathematical Sciences, University of the South Pacific, Laucala Campus, Laucala Bay Road, Suva, Central Province, Fiji
[b]Department of Applied Mathematics, Faculty of Engineering, Kobe University, Kobe, Japan

ARTICLE INFO

Article history:
Received 21 April 2007
Received in revised form 30 May 2008
Accepted 23 June 2008

Keywords:
Artificial potential fields method
Lyapunov stability
Findpath problem
Collision avoidance
Non-holonomic mobile robot

ABSTRACT

Artificial potential fields, which are widely used in robotics for path planning and collision avoidance, are normally beset by difficulties arising from the existence of local minima. This article proposes a solution that involves an asymptotically stable point-mass system governed by differential equations. The system represents a planar point robot moving from its initial position to the desired goal whilst avoiding a static obstacle. Because the system is asymptotically stable, its Lyapunov function, which produces artificial potential fields around the goal and the obstacle, has no local minima other than the goal configuration in the pathwise-connected proper subset of free space which contains the goal configuration. As an application, we consider the point stabilization of a planar mobile car-like robot moving in the presence of a static obstacle.

1. Introduction

An ongoing research in robotics involves the identification in a two- or three-dimensional space a continuous path that allows a robot, or a part of it, to reach its destination without colliding with obstacles that may exist in the space. Sometimes referred to as the *findpath problem*, it is essentially a geometric problem.

Having being analyzed over the last two decades by many researchers, it is now possible to gather a majority of the proposed solutions under two categories: (1) those that employ some kind of graph search technique and (2) those that employ some kind of physical analogy.

In a graph search technique, a collision-free path is generated by searching a graph formed out of straight lines that connect the starting position and the destination via the vertices of solid obstacles, or via patches of free space that have been decomposed into geometric primitives such as cones and cylinders. Some of the pioneering work include those in 1983 by Schwartz and Sharir [1], Brooks [2], and Lozano-Pérez [3], and that in 1986 by Herman [4]. Relatively recent applications include those by Lam and Srikanthan [5], and Williams and Jones [6] in 2001, Kruusmaa [7] and Sacks [8] in 2003, Roy [9] in 2005 and Zeghloul [10] in 2006. Theoretically, graph search techniques are elegant. However, they could involve computationally intensive algorithms. Simpler algorithms tend to use physical analogies to establish artificial potential fields with repulsive poles around obstacles and attractive poles around goals. A collision-free path is determined by how much the robot is attracted to or repelled by the poles. Pioneers in this area include Khatib [11] in 1986, Connolly et al. [12] in 1990, and Tarassenko and Blake [13], and Kim and Khosla [14] in 1991. Recent extensions and applications include those by Ge and Cui [15] in 2002, Tanner et al. [16] in 2003 and Lin et al. [17] in 2004. In any artificial potential fields method, it is a challenge, however, to construct potential fields that do not have local minima or points of zero potential and kinetic energy other than the goal configuration. Several studies have successfully considered this problem via the use of special functions. The work of Rimon and Koditschek [18] in 1992 with potential functions, and that of Tanner et al. [16] in 2003 with dipolar inverse Lyapunov functions, are noteworthy. An excellent summary of the various methods associated with artificial potential fields can be found in the work of Lee [19] in 2004.

In this article, our intention is to show the viability of directly using the well-known second method of Lyapunov to construct a Lyapunov function that ensures the asymptotic stability of an obstacle-avoidance system, and hence solves the problem of local minima. The main advantage of this *global potential approach* [19] is the ease in which it can be used to extract control laws. The proposed technique is based on the 1990 pioneering work of Stonier [20], which was later expanded and improved in 1998 and 2001 by Vanualailai et al. [21] and Ha and Shim [22], respectively. In this paper, we consider a planar obstacle-avoidance system governed by differential equations. The system consists of a point-mass being controlled to its destination or target whilst avoiding a fixed object in two-dimensional space. The proposed Lyapunov function for the system produces artificial potential fields both for obstacle-avoidance and for target attraction. After establishing Lyapunov stability, we then show that it

* Corresponding author. Tel.: +679 323 2277; fax: +679 323 1527.
E-mail address: vanualailai@usp.ac.fj (J. Vanualailai).

0020-7462/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijnonlinmec.2008.06.012

is possible to identify a region of asymptotic stability in which the target is the only minimum point. As an application, we consider the point stabilization of planar mobile robot, which is car-like and non-holonomic.

2. The Lyapunov method

Here, we briefly recall some of the important Lyapunov stability concepts that we will be using to derive our control laws.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the Euclidean norm $\|\cdot\|$. Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ denote an element of $\mathbb{R}^n$. Consider an autonomous non-linear system

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0)=\mathbf{x}_0, \ \ t_0\geqslant 0, \tag{1}$$

where $\mathbf{f}:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$ is assumed to be smooth enough to guarantee the existence, uniqueness and continuous dependence of solutions $\mathbf{x}(t)=\mathbf{x}(t;t_0,\mathbf{x}_0)$ of (1) in Ω, an open set in $\mathbb{R}^n$.

For the purpose of considering stability concept in the sense of Lyapunov, we assume there is a point $\mathbf{e}_0\in\mathbb{R}^n$ such that $\mathbf{f}(\mathbf{e}_0)\equiv\mathbf{0}$. Then $\mathbf{x}(t)\equiv\mathbf{e}_0$ is trivially a solution of (1) through $\mathbf{e}_0\in\Omega$ for all $t\geqslant t_0$. We call $\mathbf{e}_0$ an *equilibrium point* of system (1).

The equilibrium point $\mathbf{e}_0$ of (1) is *stable* if, for each $\varepsilon>0$ and $t_0\geqslant 0$, there is a $\delta=\delta(t_0,\varepsilon)>0$ such that $\|\mathbf{x}_0-\mathbf{e}_0\|<\delta$ implies $\|\mathbf{x}(t)-\mathbf{e}_0\|<\varepsilon$ for all $t\geqslant t_0$. The equilibrium point $\mathbf{e}_0$ of (1) is said to be *asymptotically stable* if it is stable and there exists $\delta(t_0)>0$ such that $\|\mathbf{x}(t_0)-\mathbf{e}_0\|<\delta$ implies $\lim_{t\to\infty}\|\mathbf{x}(t)-\mathbf{e}_0\|=0$. The equilibrium point $\mathbf{e}_0$ of (1) is said to be *globally asymptotically stable* if it is stable and $\lim_{t\to\infty}\|\mathbf{x}(t)-\mathbf{e}_0\|=0$ for all $\mathbf{x}_0\in\mathbb{R}^n$.

Lyapunov's direct method (also called the second method of Lyapunov) is summarized in the following theorem, where $\mathbb{R}^+:=[0,\infty)$:

Theorem 1. *Let $\mathbf{e}_0$ be an equilibrium point of* (1) *and let $V:\Omega\to\mathbb{R}^+$ be a C^1 function defined on some neighborhood Ω of $\mathbf{e}_0$ such that* (i) $V(\mathbf{e}_0)=0$, (ii) $V(\mathbf{x})>0$ *for* $\mathbf{x}\in\Omega\backslash\{\mathbf{e}_0\}$ *and* (iii) $\dot{V}(\mathbf{x})|_{(1)}\leqslant 0$ *for all* $\mathbf{x}\in\Omega$. *Then $\mathbf{e}_0$ is stable. If* (iii) *is replaced by* (iii)$'$ $\dot{V}(\mathbf{x})|_{(1)}<0$ *for all* $\mathbf{x}\in\Omega\backslash\{\mathbf{e}_0\}$, *then $\mathbf{e}_0$ is asymptotically stable. If $\mathbf{e}_0$ is asymptotically stable, and if, furthermore $V(\mathbf{x})$ is radially unbounded* (*that is, $V(\mathbf{x})\to\infty$ as $\|\mathbf{x}\|\to\infty$*), *then $\mathbf{e}_0$ is globally asymptotically stable.*

We refer to V in Theorem 1 as a Lyapunov function for system (1).

3. A globally asymptotically stable point-mass system

Consider a point-mass, defined as the disk of radius $r_P\geqslant 0$, and positioned at $(x(t),y(t))\in\mathbb{R}^2$ at time $t\geqslant 0$. That is, the point-mass is

$$P=\{(z_1,z_2)\in\mathbb{R}^2:(z_1-x)^2+(z_2-y)^2\leqslant r_P^2\}. \tag{2}$$

Its instantaneous velocity is $(v(t),w(t)):=(\dot{x}(t),\dot{y}(t))$. Our general ODE system is therefore of the form

$$\begin{aligned}&\dot{x}(t)=v(x(t),y(t)), \quad \dot{y}(t)=w(x(t),y(t)),\\ &(x_0,y_0):=(x(0),y(0)),\end{aligned} \tag{3}$$

and our objective is to steer the point-mass to a goal or target in $\mathbb{R}^2$. The target is defined as the disk with center (τ_1,τ_2) and radius r_T, that is,

$$T=\{(z_1,z_2)\in\mathbb{R}^2:(z_1-\tau_1)^2+(z_2-\tau_2)^2\leqslant r_T^2\}$$

with $r_T\geqslant 0$ sufficiently small. Let $\mathbf{e}_0=(\tau_1,\tau_2)$. We state our first result:

Theorem 2. *Let $v(x,y)=-(x-\tau_1)$ and $w(x,y)=-(y-\tau_2)$. Then the point $\mathbf{e}_0$ is the only equilibrium point of system* (3) *and is globally asymptotically stable.*

Proof. If $v(x,y)=-(x-\tau_1)$ and $w(x,y)=-(y-\tau_2)$, then it is clear that $\mathbf{e}_0=(\tau_1,\tau_2)$ is the only equilibrium point of the system. To prove global asymptotic stability, we use the Lyapunov function $V(x,y)=[(x-\tau_1)^2+(y-\tau_2)^2]/2$, which is radially unbounded, with $V(\mathbf{e}_0)=0$. Its time-derivative along a trajectory of system (3) is $\dot{V}_{(3)}(x,y)=-[(x-\tau_1)^2+(y-\tau_2)^2]$, with $\dot{V}_{(3)}(x,y)<0$ for all $(x,y)\neq\mathbf{e}_0$, and $\dot{V}_{(3)}(\mathbf{e}_0)=0$. □

4. An asymptotically stable point-mass system with a fixed obstacle

We next consider the situation where there is now a fixed obstacle that the point-mass P has to avoid. Precisely, if (o_1,o_2) is the center of the disk, and r_O is the radius of the disk, then the obstacle can be defined as

$$O=\{(z_1,z_2)\in\mathbb{R}^2:(z_1-o_1)^2+(z_2-o_2)^2\leqslant r_O^2\}.$$

Next, we construct an artificial potential field function that guarantees target attraction and collision avoidance.

4.1. Target attraction and collision avoidance

For target attraction, we want to have a measurement, at time $t\geqslant 0$, of the distance between the position (x,y) of the point-mass P and its target T. A likely function is therefore

$$G(x,y):=\tfrac{1}{2}[(x-\tau_1)^2+(y-\tau_2)^2].$$

For obstacle-avoidance, we want to have a measurement of the distance between the point-mass P and its obstacle O. Thus, consider the function

$$W(x,y):=\tfrac{1}{2}[(x-o_1)^2+(y-o_2)^2-(r_O+r_P)^2]. \tag{4}$$

Let us, for the moment, consider, for some constant $\alpha>0$, the effect of the ratio α/W. If P approaches the obstacle O, then W decreases and the ratio increases. Assume next that the ratio is an appropriate part of a Lyapunov function, V, that establishes the stability of an equilibrium point of system (1). Because, with respect to time $t\geqslant 0$, we have that $\mathrm{d}V/\mathrm{d}t\leqslant 0$ along a trajectory of (1), and V is a positive definite function, V cannot increase in t. Therefore any change in the value of the ratio could only correspond to either an increase or decrease in $|\mathrm{d}V/\mathrm{d}t|$. Analogously, $|\mathrm{d}V/\mathrm{d}t|$ is the rate of dissipation of energy from the system in absolute value. If the obstacle is being approached, then W gets smaller, and the ratio gets larger. Thus, the rate of energy dissipation, in absolute value, gets larger. This, in turn, results in an increased activity of the system. This increased activity could only be directed toward a stable equilibrium point, away from the obstacle. In other words, we cannot have a situation where $W=0$. Hence, if the ratio is a part of a Lyapunov function for system (1), then intuitively the ratio will act as an *obstacle-avoidance function*.

4.2. Lyapunov function as an artificial potential field function

Let $\alpha>0$ be a constant, and consider as a tentative Lyapunov function for system (3)

$$V(x,y)=G(x,y)+\frac{\alpha G(x,y)}{W(x,y)}. \tag{5}$$

It is clear that V is continuous and locally positive definite on the domain $D(V)=\{(x,y)\in\mathbb{R}^2:W(x,y)>0\}$. That is, $V(x,y)>0$ for all $(x,y)\in D(V)\backslash\{\mathbf{e}_0\}$ and $V(\mathbf{e}_0)=0$, with $\mathbf{e}_0\in D(V)$. It is clear that $D(V)$ is a pathwise-connected proper subset of $\mathbb{R}^2$, meaning that for every two points in $D(V)$, there is a path connecting them.