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Stagnation-point flow of a second-grade fluid with slip

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ABSTRACT

The steady two-dimensional stagnation-point flow of a second-grade fluid with slip is examined. The fluid impinges on the wall either orthogonally or obliquely. Numerical solutions are obtained using a quasi-linearization technique.

1. Introduction

In recent years, non-Newtonian fluids have become more and more important industrially. Polymer solutions, polymer melts, blood, paints and certain oils are the most common examples of non-Newtonian fluids. Since, Navier–Stokes equations cannot adequately describe such fluids, several non-Newtonian models were developed. Among those, the non-Newtonian second-grade fluid [1] has been studied extensively. The equations of motion of such fluids are highly non-linear and one order higher than the Navier–Stokes equations. For this reason, one will require boundary conditions in addition to the non-slip condition to have a well-posed problem. Only in some special cases where the higher-order non-linear terms in these equations can be neglected thereby reducing their order, are the "no-slip" condition sufficient to yield unique solutions. In general, Rajagopal [2], Rajagopal and Gupta [3], Rajagopal [4] and Rajagopal and Kaloni [5] have shown that the absence of this additional boundary condition leads to non-unique solutions for problems involving the flow of second-grade fluids in a bounded domain. Therefore, the "no-slip" condition is insufficient to solve the equations of motions of second-grade fluids completely when the higher-order non-linearities in these equations cannot be ignored.

One class of flows which has thoroughly been studied in literature is the stagnation-point flows. Hiemenz [6] derived an exact solution of the steady flow of a Newtonian fluid impinging orthogonally on an infinite flat plate. Stuart [7], Tamada [8] and Dorrepaal [9] independently investigated the solutions of a stagnation-point flow when the fluid impinges obliquely on the plate. Beard and Walters [10] used boundary-layer equations to study two-dimensional flow near a stagnation point of a viscoelastic fluid. Rajagopal et al. [11] have studied the Falkner–Skan flows of an incompressible second-grade fluid. Dorrepaal et al. [12] investigated the behaviour of a viscoelastic fluid impinging on a flat rigid wall at an arbitrary angle of incidence. Labropulu et al. [13] studied the oblique flow of a second-grade fluid impinging on a porous wall with suction or blowing.

In a recent paper, Wang [14] studied stagnation-point flows with slip. This problem appears in some applications where a thin film of light oil is attached to the plate or when the plate is coated with special coatings such as a thick monolayer of hydrophobic octadecylthichlorosilane [15]. Also, wall slip can occur if the working fluid contains concentrated suspensions [16].

When the molecular mean free path length of the fluid is comparable to the system's characteristic length, then rarefaction effects must be considered. The Knudsen number K_n, defined as the ratio of the molecular mean free path to the characteristic length of the system, is the parameter used to classify fluids that deviate from continuum behaviour. If $K_n > 10$, it is free molecular flow, if $0.1 < K_n < 10$ it is transition flow, if $0.01 < K_n < 0.1$ it is slip flow, and if $K_n < 0.01$ it is the viscous flow (see [14,17]). Flows in the slip-flow region have been modelled using the Navier–Stokes equations and the traditional non-slip condition is replaced by the slip condition

$$u_t = A_p \frac{\partial u_t}{\partial n} \tag{1}$$

where u_t is the tangential velocity component, n is normal to the plate, and A_p is a coefficient close to 2(mean free path)/$\sqrt{\pi}$ (see [18]). This condition was first proposed by Navier [19] nearly 200 years ago.

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Over the years, various other slip conditions have been considered. Recently, Rao and Rajagopal [20] reviewed several slip conditions and showed that the usual assumption that slip depends only on the shear stress at the wall may not be correct for some flows. They investigated linearly viscous flow in a channel and included normal stresses in describing the slip velocity at the walls. They found the solutions obtained by doing so were qualitatively different from that obtained assuming shear stresses only.

In the present work, we followed Wang [14] and considered the original slip condition as given by Navier in (1). We investigate the behaviour of the non-Newtonian second-grade fluid impinging on a rigid wall with slip. The fluid impinges on the wall either orthogonally or obliquely. In particular, we study the effects of the slip condition and the effects of the viscoelasticity of the fluid on the stagnation-point.

2. Flow equations

The flow of a viscous incompressible non-Newtonian second-grade fluid, neglecting thermal effects and body forces, is governed by

$$\operatorname{div} \underset{\sim}{V}^* = 0 \tag{2}$$

$$\rho \dot{\underset{\sim}{V}}^* = \operatorname{div} \underset{\approx}{T} \tag{3}$$

when the constitutive equation for the Cauchy stress tensor $\underset{\approx}{T}$ which describes second-grade fluids given by Rivlin and Ericksen [1] is

$$\left.\begin{aligned} \underset{\approx}{T} &= -p^* \underset{\approx}{I} + \mu \underset{\approx}{A}_1 + \alpha_1 \underset{\approx}{A}_2 + \alpha_2 \underset{\approx}{A}_1^2 \\ \underset{\approx}{A}_1 &= (\operatorname{grad} \underset{\sim}{V}^*) + (\operatorname{grad} \underset{\sim}{V}^*)^{\mathrm{T}} \\ \underset{\approx}{A}_2 &= \dot{\underset{\approx}{A}}_1 + (\operatorname{grad} \underset{\sim}{V}^*)^{\mathrm{T}} \underset{\approx}{A}_1 + \underset{\approx}{A}_1 (\operatorname{grad} \underset{\sim}{V}^*) \end{aligned}\right\} \tag{4}$$

where $\underset{\sim}{V}^*$ is the velocity vector field, p^* the fluid pressure function, ρ the constant fluid density, μ the constant coefficient of viscosity and α_1, α_2 the normal stress moduli.

For a two-dimensional flow, we take $\underset{\sim}{V}^* = (u^*(x^*, y^*), v^*(x^*, y^*))$ and $p^* = p^*(x^*, y^*)$ so that the flow equations (2)–(3) become

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{5}$$

$$\begin{aligned} u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + \frac{1}{\rho}\frac{\partial p^*}{\partial x^*} &= \nu \nabla^{*2} u^* + \frac{\alpha_1}{\rho} \left\{ \frac{\partial}{\partial x^*} \left[2u^* \frac{\partial^2 u^*}{\partial x^{*2}} + 2v^* \frac{\partial^2 u^*}{\partial x^* \partial y^*} + 4\left(\frac{\partial u^*}{\partial x^*}\right)^2 \right.\right. \\ &\quad \left. + 2\frac{\partial v^*}{\partial x^*}\left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right)\right] + \frac{\partial}{\partial y^*}\left[\left(u^* \frac{\partial}{\partial x^*} + v^* \frac{\partial}{\partial y^*}\right)\right. \\ &\quad \left.\left. \times \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right) + 2\frac{\partial u^*}{\partial x^*}\frac{\partial u^*}{\partial y^*} + 2\frac{\partial v^*}{\partial x^*}\frac{\partial v^*}{\partial y^*}\right]\right\} \\ &\quad + \frac{\alpha_2}{\rho}\frac{\partial}{\partial x^*}\left[4\left(\frac{\partial u^*}{\partial x^*}\right)^2 + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right)^2\right] \end{aligned} \tag{6}$$

$$\begin{aligned} u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + \frac{1}{\rho}\frac{\partial p^*}{\partial y^*} &= \nu \nabla^{*2} v^* + \frac{\alpha_1}{\rho} \left\{ \frac{\partial}{\partial x^*} \left\{ \left[2\frac{\partial v^*}{\partial x^*}\frac{\partial v^*}{\partial y^*} + \left(u^* \frac{\partial}{\partial x^*} + v^* \frac{\partial}{\partial y^*}\right)\left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right) \right.\right.\right. \\ &\quad \left. + 2\frac{\partial v^*}{\partial x^*}\frac{\partial v^*}{\partial y^*}\right] + \frac{\partial}{\partial y^*}\left[2u^* \frac{\partial^2 v^*}{\partial x^* \partial y^*} + 4\left(\frac{\partial v^*}{\partial y^*}\right)^2 + 2v^* \frac{\partial^2 v^*}{\partial x^{*2}} \right. \\ &\quad \left.\left.\left. + 2\frac{\partial u^*}{\partial y^*}\left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right)\right]\right\}\right\} + \frac{\alpha_2}{\rho}\frac{\partial}{\partial y^*}\left[4\left(\frac{\partial v^*}{\partial y^*}\right)^2 \right. \\ &\quad \left. + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right)^2\right] \end{aligned} \tag{7}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. The star on a variable indicates its dimensional form. Using the non-dimensional variables

$$x = x^* \sqrt{\frac{\beta}{\nu}}, \quad y = y^* \sqrt{\frac{\beta}{\nu}}$$
$$u = \frac{1}{\sqrt{\nu\beta}} u^*, \quad v = \frac{1}{\sqrt{\nu\beta}} v^*, \quad p = \frac{1}{\rho\nu\beta} p^*$$

where β has the units of inverse time, the flow equations take the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8}$$

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= \nabla^2 u + W_e \left\{ \frac{\partial}{\partial x}\left[2u\frac{\partial^2 u}{\partial x^2} + 2v\frac{\partial^2 u}{\partial x \partial y} + 4\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \right. \\ &\quad \left. + \frac{\partial}{\partial y}\left[\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]\right\} \\ &\quad + \lambda\frac{\partial}{\partial x}\left[4\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right] \end{aligned} \tag{9}$$

$$\begin{aligned} u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} &= \nabla^2 v + W_e \left\{ \frac{\partial}{\partial x}\left[\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right] \right. \\ &\quad \left. + \frac{\partial}{\partial y}\left[2u\frac{\partial^2 v}{\partial x \partial y} + 2v\frac{\partial^2 v}{\partial y^2} + 4\left(\frac{\partial v}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]\right\} \\ &\quad + \lambda\frac{\partial}{\partial y}\left[4\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right] \end{aligned} \tag{10}$$

where $W_e = \alpha_1\beta/\rho\nu$ is the Weissenberg number and $\lambda = \alpha_2\beta/\rho\nu$.

Using continuity equation (8), we define the streamfunction $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{11}$$

Substitution of (11) in Eqs. (9) and (10) and elimination of pressure from the resulting equations using $p_{xy} = p_{yx}$ yields

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} - W_e \frac{\partial(\psi, \nabla^4\psi)}{\partial(x, y)} + \nabla^4\psi = 0 \tag{12}$$

Having obtained a solution of Eq. (12), the velocity components are given by (11) and the pressure can be found by integrating Eqs. (9) and (10).