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Non-linear in-plane buckling of rotationally restrained shallow arches under a central concentrated load

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Abstract

This paper investigates the non-linear in-plane buckling of pin-ended shallow circular arches with elastic end rotational restraints under a central concentrated load. A virtual work method is used to establish both the non-linear equilibrium equations and the buckling equilibrium equations. Analytical solutions for the non-linear in-plane symmetric snap-through and antisymmetric bifurcation buckling loads are obtained. It is found that the effects of the stiffness of the end rotational restraints on the buckling loads, and on the buckling and postbuckling behaviour of arches, are significant. The buckling loads increase with an increase of the stiffness of the rotational restraints. The values of the arch slenderness that delineate its snap-through and bifurcation buckling modes, and that define the conditions of buckling and of no buckling for the arch, increase with an increase of the stiffness of the rotational end restraints.

Keywords: Analysis; Arches; Bifurcation; Buckling: Closed form solutions; Elastic; Instability; Non-linear; Restraints; Rotational; Shallow; Snap-through

1. Introduction

A shallow arch that is fully braced laterally and that is subjected to in-plane loading (Fig. 1) may buckle in an in-plane antisymmetric bifurcation mode or in a symmetric snap-through mode (Fig. 2). It has been found [1] that the structural behaviour of the arch becomes quite non-linear before in-plane buckling, and so the effects of this non-linearity on the in-plane buckling of shallow arches need to be considered. The buckling of sinusoidal shallow arches was studied by Timoshenko and Gere [2] and Simitses [3]. Gjelsvik and Bodner [4] used an energy method to investigate the instability of fixed shallow circular arches with rectangular solid cross-section subjected to central point loading, and approximate solutions were obtained. Schreyer and Masur [5] performed an exact analysis for shallow circular arches and derived analytical solutions, but their analysis was limited to fixed arches with a rectangular solid section. Dickie and Broughton [6] used a series method to study the buckling of shallow circular pin-ended and fixed arches. However, their study was also confined to rectangular solid cross-sections and only approximate numerical solutions were obtained. Pi et al. [7] studied the in-plane non-linear buckling of circular arches of arbitrary cross-section that are subjected to a radial load distributed uniformly around the arch axis, while Bradford et al. [8] investigated the in-plane non-linear buckling of circular arches of arbitrary cross-section that are subjected to a central concentrated radial load.

In practice, however, the ends of an arch are not necessarily fixed or pin-ended. In many cases, an arch may be supported by elastic foundations or by other structural elements that provide elastic rotational restraint at its ends. These restraints can be replaced by equivalent elastic rotational springs and the arch can be considered to be supported elastically at the ends by these springs. By knowing the structural configuration of the arch supports, the stiffness of the corresponding springs (or the spring constant) can often be accurately estimated. These end restraints participate in the structural behaviour of the arch and may influence significantly its in-plane buckling and postbuckling response. In order to investigate the buckling and postbuckling behaviour of elastically restrained shallow arches, both the effects of the prebuckling non-linear deformations and the stiffness of the elastic restraints have to be

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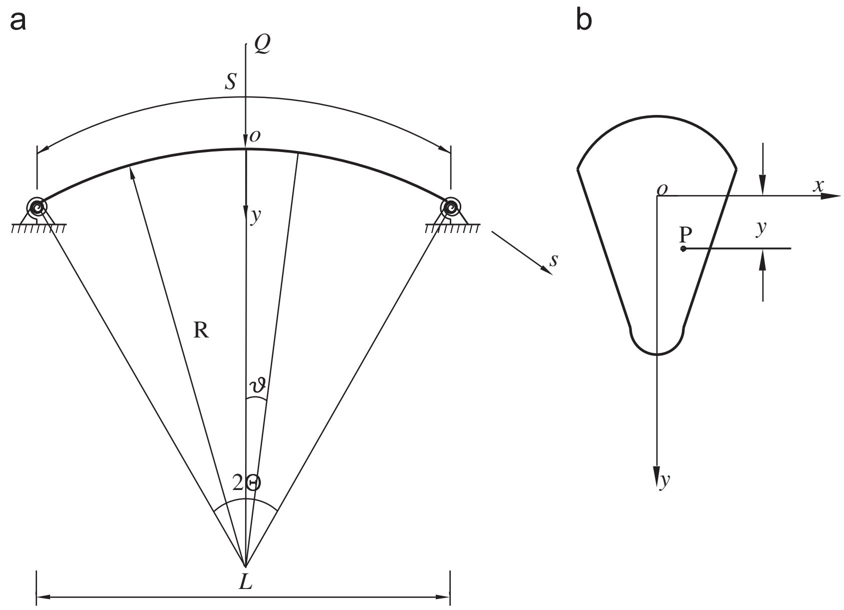

Fig. 1. Arch model.

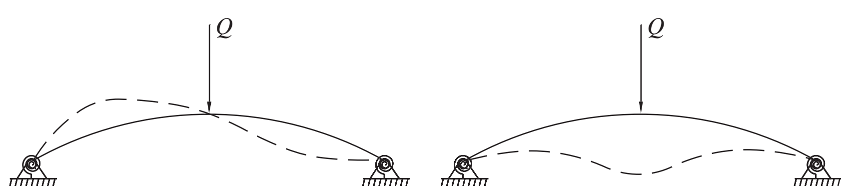

Fig. 2. Antisymmetric and symmetric buckling.

considered. Studies of the in-plane buckling of arches with elastic rotational end restraints do not appear to be reported in the open literature, and it is not known how the elastic supports affect the buckling and postbuckling behaviour of an arch, nor under what circumstances antisymmetric bifurcation buckling or symmetric snap-through buckling occurs under influence of the end elastic restraints.

The purpose of this paper is to investigate the in-plane buckling of circular shallow pin-ended arches with equal elastic rotational restraints at both ends (Fig. 1) that are subjected to a central concentrated radial load, by using a virtual work technique to establish the non-linear equilibrium equations. These equations are then used to obtain analytical solutions for the non-linear bifurcation and snap-through buckling loads, to determine the slenderness limits for distinguishing the buckling modes, and to investigate the buckling and postbuckling behaviour.

2. Non-linear in-plane equilibrium

2.1. Differential equations of equilibrium

A virtual work formulation is used to establish the conditions for the non-linear relationship between the central concentrated load and the internal axial compressive force in an arch. The longitudinal normal strain of a point *P* can expressed as

$$\varepsilon = \varepsilon_{\mathrm{m}} + \varepsilon_{\mathrm{b}}, \tag{1}$$

where ε_{m} and ε_{b} are the membrane and bending strains, respectively, and are given by (Pi et al., 2002)

$$\varepsilon_{\mathrm{m}} = \tilde{w}' - \tilde{v} + \frac{1}{2}(\tilde{v}')^2 \tag{2}$$

and

$$\varepsilon_{\mathrm{b}} = -\frac{y\tilde{v}''}{R}, \tag{3}$$

in which $(\ )' \equiv \mathrm{d}(\ )/\mathrm{d}\theta$, θ is the angular coordinate; $\tilde{v} = v/R$, $\tilde{w} = w/R$, v and w are the radial and axial displacements, respectively, R is the radius of initial curvature of the arch, and y is the coordinate of the point P in the principal axis oy (Fig. 1).