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On sheet-driven motion of power-law fluids

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Abstract

A rigorous analysis of non-Newtonian boundary layer flow of power-law fluids over a stretching sheet is presented. First, a systematic framework for treatment of sheet velocities of the form $U(x)=Cx^m$ is provided. By means of an exact similarity transformation, the non-linear boundary layer momentum equation transforms into an ordinary differential equation with m and the power-law index n as the only parameters. Earlier investigations of a continuously moving surface ($m=0$) and a linearly stretched sheet ($m=1$) are recovered as special cases.

For the particular parameter value $m=1$, i.e. linear stretching, numerical solutions covering the parameter range $0.1 \leqslant n \leqslant 2.0$ are presented. Particular attention is paid to the most shear-thinning fluids, which exhibit a challenging two-layer structure. Contrary to earlier observations which showed a monotonic decrease of the sheet velocity gradient $-f''(0)$ with n, the present results exhibit a local minimum of $-f''(0)$ close to $n=1.77$. Finally, a series expansion in $(n-1)$ is proved to give good estimates of $-f''(0)$ both for shear-thinning and shear-thickening fluids.

Keywords: Boundary layer theory; Non-Newtonian fluids; Similarity solutions

1. Introduction

Non-linear fluid rheology is encountered in numerous practical situations and the study of non-Newtonian fluid motion is accordingly an important subset of fluid mechanics. Among the most popular rheological models for non-Newtonian fluids is the power-law or Ostwald–deWaele model. This model is a simple *non-linear* equation of state for inelastic fluids which includes *linear* Newton-fluids as a special case. The power-law model provides an adequate representation of many non-Newtonian fluids over the most important range of shear rates. This, together with its apparent simplicity, has made it a very attractive model both in analytical and numerical research. An account of the earlier investigations is provided in the introduction to the computational study by Andersson and Toften [1], while recent analytical considerations by Denier and Dabrowski [2] addressed some of the mathematical subtleties associated with the power-law model.

A particular class of flow problems, which has received considerable attention over the years, is the fluid flow driven by the motion of a flat surface. The moving surface, which might be a polymer sheet or an extruded filament, emerges from a slit, from which a viscous boundary layer flow develops in the direction of the moving surface. This problem was first analyzed by Sakiadis [3,4] who studied the boundary layer flow driven by a continuously moving sheet, whereas Crane [5] considered the corresponding problem of flow induced by a linearly stretched surface. Thereafter Afzal and Varshney [6] and Banks [7] independently provided a unified analysis for more general sheet boundary conditions, which included Sakiadis' and Crane's solutions as special cases. Banks [7] also pointed out some earlier works that arrived to the same governing equation as that obtained for sheet-driven fluid motion, but in a rather different context.

This class of flow problems, notably the Sakiadis boundary layer and the Crane boundary layer, has been extended to fluids exhibiting non-Newtonian rheology. While Chiam [8] considered the motion of a micropolar fluid over a stretching sheet, the corresponding flow of elastico-viscous fluids was studied

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by Siddappa and Khapate [9], Rajagopal et al. [10], Siddappa and Abel [11] and Maneschy et al. [12]. These and more recent studies rely on boundary-layer-type approximations in order to simplify the Cauchy equation. The applicability of the conventional boundary layer concept to non-Newtonian fluids has been addressed by Bizzel and Slattery [13], Astin et al. [14], Rajagopal et al. [15] and more recently by Denier and Dabrowski [2].

We now confine ourselves to the flow of inelastic power-law fluids driven by a flat surface. Following for instance Mansutti and Rajagopal [16], the constitutive equation for a power-law fluid can be expressed as

$$\boldsymbol{T} = -p\boldsymbol{I} + K(tr\boldsymbol{A}^2)^M\boldsymbol{A}. \tag{1}$$

Here, the Cauchy stress tensor $\boldsymbol{T}$ is expressed in terms of the pressure p, the material constants K and M and the identity matrix $\boldsymbol{I}$, while the first Rivlin–Ericksen tensor $\boldsymbol{A}$ is defined in terms of the velocity vector $\boldsymbol{u}$ as

$$\boldsymbol{A} = (\text{grad } \boldsymbol{u}) + (\text{grad } \boldsymbol{u})^{\mathrm{T}}. \tag{2}$$

Newtonian (i.e. linear) rheology is recovered for $M = 0$, while positive and negative M-values correspond to shear-thickening and shear-thinning fluids, respectively. The analysis of Sakiadis [3,4] of Newtonian flow over a continuously moving sheet was extended to power-law fluids by Fox et al. [17], while Crane's [5] work on a Newtonian flow driven by a linearly stretching sheet was extended by Andersson and Dandapat [18] to power-law fluids. Yürüsoy [19] very recently studied the unsteady flow of a power-fluid driven by a linearly stretched sheet where the stretching rate decreased with time.

In the present paper, we first aim to provide a unified analysis of non-Newtonian fluid flow driven by a steadily moving plane sheet with a surface velocity proportional to the distance from the slit raised to an arbitrary power m. New numerical results for the particular parameter value $m = 1$, covering a wider range of the power-law index $n = 2M + 1$, will be presented herein, together with accurate series solutions for n close to unity and an asymptotic solution for the highly non-linear parameter range $n < 0.5$.

2. Problem formulation

2.1. Governing equations of motion

We consider the steady and two-dimensional flow governed by the boundary layer equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial T_{xy}}{\partial y}, \tag{4}$$

which assure conservation of mass and streamwise momentum, respectively. Here, u and v denote the velocity components in the streamwise (x) and the cross-stream (y) directions, respectively. The power-law fluid is represented by the rheological equation of state (1), which in Cartesian tensor notation becomes

$$T_{ij} = -p\delta_{ij} + 2K(D_{kl}D_{kl})^{(n-1)/2} \cdot D_{ij}, \tag{5}$$

where T_{ij} and D_{ij} are the stress and strain rate tensors, K and n are the consistency coefficient and power-law index and ρ is the fluid density. It is noteworthy that the power-law index n is related to the exponent M in Eq. (1) as $n = 2M + 1$. The constitutive equation (5) represents shear-thinning (pseudoplastic) fluids for $n < 1$ and shear-thickening (dilatant) fluids for $n > 1$, whereas $n = 1$ corresponds to Newtonian (i.e. linear) rheology.

Within the boundary layer approximation, the essential off-diagonal stress component simplifies to

$$T_{xy} = K\left(\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial y}\right)^{(n-1)/2}\cdot\frac{\partial u}{\partial y} = -K\left(-\frac{\partial u}{\partial y}\right)^n \tag{6}$$

and becomes the only stress component of dynamic significance. The governing momentum equation (4) can therefore be written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{K}{\rho}\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^n. \tag{7}$$

Here, and on the right-hand side of Eq. (6), the shear rate $\partial u/\partial y$ has been assumed to be negative throughout the entire boundary layer since the streamwise velocity component u decreases monotonically with the distance y from the moving surface.

A rigorous derivation and subsequent analysis of the boundary layer equations for power-law fluids were recently provided by Denier and Dabrowski [2]. They focused on boundary layer flow driven by a freestream $U(x) \sim x^m$, i.e. of the Falkner–Skan type. Such boundary layer flows are driven by a streamwise pressure gradient $-\mathrm{d}P/\mathrm{d}x = \rho U\,\mathrm{d}U/\mathrm{d}x$ set up by the external (inviscid) freestream outside the viscous boundary layer. In the present context, however, no driving pressure gradient is present. Instead, the flow is driven solely by a flat surface which moves with a prescribed velocity $U(x) = Cx^m$, where x denotes the distance from the slit from which the surface emerges and $C(>0)$ and m are constants. The relevant boundary conditions for the problem at hand thus become:

$$u(x,0) = Cx^m, \tag{8a}$$

$$v(x,0) = 0, \tag{8b}$$

$$u \to 0 \quad \text{as } y \to \infty. \tag{8c}$$

While (8c) ascertains that the fluid velocity vanishes outside the boundary layer, the requirement (8b) signifies impermeability of the stretching surface positioned at $y = 0$, whereas (8a) assures no-slip at the surface, i.e. zero velocity difference between the fluid and the surface. Besides the boundary conditions (8), the auxiliary requirement that the shear stress T_{xy} should vanish outside the momentum boundary layer needs to be satisfied by a proper solution. This latter requirement applies both for Newtonian $(n = 1)$ and non-Newtonian $(n \neq 1)$ fluids and its importance was recently addressed by Andersson and Aarseth [20].