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Original Article

Nonlinear transmission conditions for thin highly conductive interphases of curvilinear shape

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ABSTRACT

We consider heat transfer problem in a composite ceramic featuring a thin nonlinear interphase layer with distinctively different characteristics (high thermal conductivity, apart from the mentioned physical size). The presence of an interphase may be problematic for the classical FEM approach in terms of technical implementation, accuracy and stability of the results. We avoid the potential issues by replacing the interphase in the model with a zero thickness imperfect nonlinear interface with two transmission conditions. These conditions are carefully derived using asymptotic analysis and aim at preserving the physical properties of the original interphase layer now absent in the model, thus ensuring an accurate solution. Numerical examples with particular attention to various physical and geometrical aspects illustrate the validity of the described approach.

1. Introduction

Composite materials are characterised by enhanced physical properties due to their structure [11,27,14–16,33,13]. These account for the widespread use composites have gained in the past decades in every possible area of application, from energy production [1,6] to civil construction [29,35] to electronics [41] to automotive, aeronautic and aerospace engineering [37,40]. Many of these areas of technology are particularly conscious of safety-related issues, which makes accuracy a primary target while creating models of the composites in use. Unfortunately, the straightforward use of the classical finite element approach to modelling materials with small structural features, such as thin layers called *interphases*, often leads to undesired results, such as unrealistic and inaccurate solutions or numerical instability [32,26,30,31,28]. This explains the need to introduce alternative ways to model such composites.

Depending on the priorities, there are several directions to take. For example, one may attempt to give a new formulation of the finite elements, as was done for the problem of heat transfer in a composite with a thin conductive interphase in [32]. This resulted in reducing the degrees of freedom and achieving faster construction of mesh and computation in comparison with the classical approach. Yet a limitation of this method was that it was not possible to have a detailed solution within the interphase region.

Another way is to simplify the structure, i.e. to replace the thin layer in the model with an object of zero thickness. For instance, Lebon and Rizzoni do so by means of a two-level model with a perfect contact interface at the first level and imperfect interface at the second one [20,21,36,37]. More commonly, however, the interphase layer is represented in the model as an imperfect interface with a set of transmission conditions that simulate the physical behaviour of the original interphase [4,5,24,22,23,26,42]. Having developed this method for low-conductive curvilinear layers in [2,3], we moved on to apply a similar approach for the situation when the interphase is, contrarily, highly conductive [39]. Such a setting is similar to considering composites with stiff interphases [10,25]. The case of a highly conductive interphasial layer can be encountered, for example, in metal reinforced ceramics that have been the object of a variety of studies regarding their thermal conductivity, increased toughness and other physical parameters and processes as well as the methods of obtaining such materials [9,18,19,34,38].

It is worth mentioning that interfacial energy is often introduced within the models with zero-thickness interface [7,8,12,17]. These works show the significant influence of the structure of the interface, and, particularly, the way it may affect wave propagation, including surface waves.

In the next section of this paper we formulate the considered problem and obtain the transmission conditions that we intend for use. Section 3 in its turn provides the numerical examples to support our analytical results. The final section collects the drawn conclusions and the scope set for future work.

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2. Problem formulation and the derivation of transmission conditions

We are considering heat transfer in a cylindrical composite with a thin highly conductive interphase of a shape close to a ring. To be precise, the boundaries are smooth closed curves of small curvature, while the centre line in a circle $r_0(\phi) = r_0$.

Initially we are trying to solve the heat transfer equation with conditions of perfect contact with the interphase:

$$\nabla\cdot(k\nabla T) + Q = c\rho\frac{\partial T}{\partial t}, \tag{1}$$

$$[T]|_{\Gamma_\pm} = 0, \tag{2}$$

$$[\mathbf{nq}]|_{\Gamma_\pm} = 0. \tag{3}$$

Fourier's law for defining the heat flux is $q = -k(T)\nabla T$.

After switching to polar coordinates (see (1) and (2)), the boundary value problem is transformed into

$$\begin{aligned}&\tfrac{1}{r}\tfrac{\partial}{\partial r}\left(\mathrm{kr}\tfrac{\partial T}{\partial r}\right) + \tfrac{1}{r^2}\tfrac{\partial}{\partial \phi}\left(k\tfrac{\partial T}{\partial \phi}\right) + Q(\phi, T) = c\rho\tfrac{\partial T}{\partial t},\\ &T_\pm - T(r_\pm, \phi, t) = 0,\\ &q_\pm + n_r^\pm k\tfrac{\partial}{\partial r}T(r_\pm, \phi, t) + n_\phi^\pm k\tfrac{1}{r_\pm}\tfrac{\partial}{\partial \phi}T(r_\pm, \phi, t) = 0.\end{aligned} \tag{4}$$

For deriving the transmission conditions in this case, we follow our approach [3,2] of rescaling the interphase by means of rescaling its width:

$$\widetilde{h} = \frac{1}{\varepsilon}h, \tag{5}$$

where $\widetilde{h}$ becomes proportional to the sizes of the adjacent layers. At the same time, we rescale the heat source and the thermal conductivity of the interphase material

$$\widetilde{Q}(T, r, \phi) = \varepsilon Q(T, r, \phi), \quad \widetilde{k}(T, r, \phi) = \varepsilon k(T, r, \phi). \tag{6}$$

This makes the interphase characteristics comparable in value with the other parameters.

Throughout this procedure, we introduce the new coordinate

$$\xi = \frac{r - r_0}{\varepsilon\widetilde{h}(\phi)}, \tag{7}$$

and the normal vectors to the boundaries are, therefore,

$$n_\pm = [n_\xi^\pm, n_\phi^\pm] = \frac{[r_0(\phi) \pm 1/2\varepsilon\widetilde{h}(\phi), \pm 1/2\varepsilon\widetilde{h}'(\phi)]}{\sqrt{(r_0(\phi) \pm 1/2\varepsilon\widetilde{h}(\phi))^2 + 1/4\varepsilon^2(\widetilde{h}'(\phi))^2}}, \tag{8}$$

where the constant centre line, i.e. $r_0' = 0$, has been taken into account.

We also redefine the temperature as a function of the new variable,

$$\widetilde{T}(\xi, \phi, t) = T(r, \phi, t), \tag{9}$$

and, as follows from (7) and (9),

$$\frac{\partial T}{\partial r} = \frac{1}{\varepsilon\widetilde{h}(\phi)}\frac{\partial\widetilde{T}}{\partial\xi}, \tag{10}$$

$$\frac{\partial T}{\partial\phi} = \frac{\partial\widetilde{T}}{\partial\phi} - \frac{r_0'(\phi) + \varepsilon\xi\widetilde{h}'(\phi)}{\varepsilon\widetilde{h}(\phi)}\frac{\partial\widetilde{T}}{\partial\xi}. \tag{11}$$

The described transformations bring (24)(24_1) to the view

$$\begin{aligned}&\tfrac{1}{(r_0+\varepsilon\xi\widetilde{h})^2}\tfrac{1}{\varepsilon}\left(\tfrac{\partial}{\partial\phi} - \tfrac{\xi\widetilde{h}'(\phi)}{\widetilde{h}(\phi)}\tfrac{\partial}{\partial\xi}\right)\left(\widetilde{k}\left(\tfrac{\partial\widetilde{T}}{\partial\phi} - \tfrac{\xi\widetilde{h}'(\phi)}{\widetilde{h}(\phi)}\tfrac{\partial\widetilde{T}}{\partial\xi}\right)\right)+\\ &\tfrac{1}{(r_0+\varepsilon\xi\widetilde{h})}\tfrac{1}{\varepsilon^3\widetilde{h}^2}\tfrac{\partial}{\partial\xi}\left(\widetilde{k}(r_0 + \varepsilon\xi\widetilde{h})\tfrac{\partial\widetilde{T}}{\partial\xi}\right) + \tfrac{1}{\varepsilon}\widetilde{Q} = c\rho\tfrac{\partial\widetilde{T}}{\partial t}.\end{aligned} \tag{12}$$

Here and henceforth, for the sake of brevity, we are omitting the arguments of the functions, bearing, however, in mind that in reality $\widetilde{h}, \widetilde{k}, \widetilde{Q}$ are all, generally speaking, non-constant.

While the form of the first condition (4)(4_2) is obvious also in the new terms, the second one (4)(4_3) transforms into

$$q_\pm + n_\xi^\pm\frac{1}{\varepsilon^2}\frac{\widetilde{k}}{\widetilde{h}}\frac{\partial}{\partial\xi}\widetilde{T}(\pm\frac{1}{2}, \phi, t) + n_\phi^\pm\frac{(\pm 2)\widetilde{k}}{\varepsilon}\left(\frac{\pm\frac{1}{2}\widetilde{h}'}{\widetilde{h}}\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\phi}\right)\widetilde{T}(\pm\frac{1}{2}, \phi, t) = 0. \tag{13}$$

We now look for a solution to this boundary value problem in the asymptotic form

$$\widetilde{T} = \widetilde{T}_0 + \varepsilon\widetilde{T}_1 + \varepsilon^2\widetilde{T}_2 + O(\varepsilon^3). \tag{14}$$

We shall also make use of the expansion of the normal vector

$$n_\pm = n_0 \pm \varepsilon n_1 + \varepsilon^2 n_2 + O(\varepsilon^3), \tag{15}$$

where

$$n_0 = [1; 0], \quad n_1 = \left[0; \frac{\widetilde{h}'}{2r_0}\right], \tag{16}$$

and of the coefficients

$$\frac{1}{(r_0 + \varepsilon\xi\widetilde{h})} = \frac{1}{r_0} - \frac{\xi\widetilde{h}}{r_0^2}\varepsilon + \frac{\xi^2\widetilde{h}^2}{r_0^3}\varepsilon^2 + O(\varepsilon^3), \tag{17}$$

$$\frac{1}{(r_0 + \varepsilon\xi\widetilde{h})^2} = \frac{1}{r_0^2} - \frac{2\xi\widetilde{h}}{r_0^3}\varepsilon + O(\varepsilon^2). \tag{18}$$

Now let us substitute the asymptotic expansions into the boundary value problem. The governing equation will at this step be represented as a series of equations grouped by the powers of ε:

$$\varepsilon^{-3}: \frac{\partial}{\partial\xi}(\widetilde{k}\frac{\partial\widetilde{T}_0}{\partial\xi}) = 0, \tag{19}$$

$$\varepsilon^{-2}: \frac{1}{\widetilde{h}}\frac{\partial}{\partial\xi}(\widetilde{k}\frac{\partial\widetilde{T}_1}{\partial\xi}) - \xi\frac{\partial}{\partial\xi}(\widetilde{k}\frac{\partial\widetilde{T}_0}{\partial\xi}) + \frac{\partial}{\partial\xi}(\xi\widetilde{k}\frac{\partial\widetilde{T}_0}{\partial\xi}) = 0, \tag{20}$$

$$\begin{aligned}&\varepsilon^{-1}: \tfrac{\xi^2}{r_0^2}\tfrac{\partial}{\partial\xi}(\widetilde{k}\tfrac{\partial\widetilde{T}_0}{\partial\xi}) - \tfrac{\xi}{r_0^2\widetilde{h}}\tfrac{\partial}{\partial\xi}(\widetilde{k}\xi\widetilde{h}\tfrac{\partial\widetilde{T}_0}{\partial\xi} + \widetilde{k}r_0\tfrac{\partial\widetilde{T}_1}{\partial\xi})+\\ &\tfrac{1}{r_0^2}\left(\tfrac{\partial}{\partial\phi}(\widetilde{k}\tfrac{\partial\widetilde{T}_0}{\partial\phi}) - \tfrac{\partial}{\partial\phi}(\tfrac{\xi\widetilde{h}'}{\widetilde{h}}\widetilde{k}\tfrac{\partial\widetilde{T}_0}{\partial\xi}) - \tfrac{\xi\widetilde{h}'}{\widetilde{h}}\tfrac{\partial}{\partial\xi}(\widetilde{k}\tfrac{\partial\widetilde{T}_0}{\partial\phi}) + \tfrac{\xi\widetilde{h}'}{\widetilde{h}}\tfrac{\partial}{\partial\xi}(\tfrac{\xi\widetilde{h}'}{\widetilde{h}}\widetilde{k}\tfrac{\partial\widetilde{T}_0}{\partial\xi})\right)+\\ &\tfrac{1}{r_0\widetilde{h}^2}\tfrac{\partial}{\partial\xi}(\widetilde{k}\xi\widetilde{h}\tfrac{\partial\widetilde{T}_1}{\partial\xi} + \widetilde{k}r_0\tfrac{\partial\widetilde{T}_2}{\partial\xi}) + \widetilde{Q} = 0.\end{aligned} \tag{21}$$

We note that upon substitution of the normal vector expansion,

$$q_\pm + \frac{1}{\varepsilon^2}\frac{\widetilde{k}}{\widetilde{h}}\frac{\partial}{\partial\xi}\widetilde{T}(\pm\frac{1}{2}, \phi, t) + \frac{\widetilde{k}\widetilde{h}'}{r_0}\left(\frac{\pm\frac{1}{2}\widetilde{h}'}{\widetilde{h}}\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\phi}\right)\widetilde{T}(\pm\frac{1}{2}, \phi, t) = 0. \tag{22}$$

Therefore, we get in the leading terms the following boundary value problem

$$\begin{aligned}&\tfrac{\partial}{\partial\xi}\left(\widetilde{k}\tfrac{\partial\widetilde{T}_0}{\partial\xi}\right) = 0,\\ &T_\pm - \widetilde{T}_0(\pm\tfrac{1}{2}, \phi, t) = 0,\\ &\widetilde{k}\tfrac{\partial\widetilde{T}_0}{\partial\xi}|_{(\pm\frac{1}{2},\phi,t)} = 0.\end{aligned} \tag{23}$$

It can be easily noticed from $\widetilde{k}\frac{\partial\widetilde{T}_0}{\partial\xi} = 0$ and $\frac{\partial\widetilde{T}_0}{\partial\xi}|_{(\pm\frac{1}{2},\phi,t)} = 0$ that $\widetilde{T}_0(\xi, \phi, t) = \widetilde{T}_0(\phi, t)$, i.e. in the leading terms the temperature is constant with respect to the coordinate ξ. At the same time (23)(23_2) is already sufficient to be used as the first transmission condition. The further analysis we are conducting to obtain just the second transmission condition.

In terms of $\widetilde{T}_1$, the boundary value problem is identical to (23), which brings us to the analogous conclusion that $\widetilde{T}_1(\xi, \phi, t) = \widetilde{T}_1(\phi, t)$, or, again, there is no dependence on ξ.

We now consider the boundary value problem in terms of $\widetilde{T}_2$,