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On an attempt to simplify the Quartapelle–Napolitano approach to computation of hydrodynamic forces in open flows

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Abstract

In this paper we are interested in the Quartapelle–Napolitano approach to calculation of forces in viscous incompressible flows in exterior domains. We study the possibility of deriving a simpler formulation of this approach which might lead to a more convenient expression for the hydrodynamic force, but conclude that such a simplification is, within the family of approaches considered, impossible. This shows that the original Quartapelle–Napolitano formula is in fact "optimal" within this class of approaches.

Keywords: Navier–Stokes equation; Hydrodynamic forces; Variational methods

1. Introduction

Calculation of hydrodynamic forces acting on an object immersed in a fluid is one of the central objectives in many applied problems in Fluid Dynamics. In this investigation we analyse the possibility of extending the approach to calculation of forces proposed by Quartapelle and Napolitano (1983). We will be concerned with incompressible flows in unbounded exterior domains (Fig. 1(a)). In some derivations we will also consider truncations Ω_1 of the domain Ω obtained by imposing an exterior boundary Γ_1 (Fig. 1(b)). We will fix the origin of the coordinate system at the obstacle and will assume that the obstacle remains motionless with the fluid velocity vanishing on its boundary. We will also assume that there is a uniform flow $U_\infty \mathbf{e}_1$ at infinity ($\mathbf{e}_1$ is the unit vector corresponding to the OX axis). The fluid motion is governed by the Navier–Stokes system representing conservation of mass and momentum. This system of equations will be assumed to have the following form:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} + \boldsymbol{\nabla}\frac{\mathbf{u}^2}{2} + \boldsymbol{\nabla} p + \nu \boldsymbol{\nabla} \times \boldsymbol{\omega} = 0 \quad \text{in } \Omega \times [0, T], \tag{1a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times [0, T], \tag{1b}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \tag{1c}$$

$$\mathbf{u}|_{\Gamma_0} = 0 \quad \text{in } [0, T], \tag{1d}$$

$$\mathbf{u} \longrightarrow U_\infty \mathbf{e}_1 \quad \text{in } [0, T] \text{ for } |\mathbf{x}| \to \infty, \tag{1e}$$

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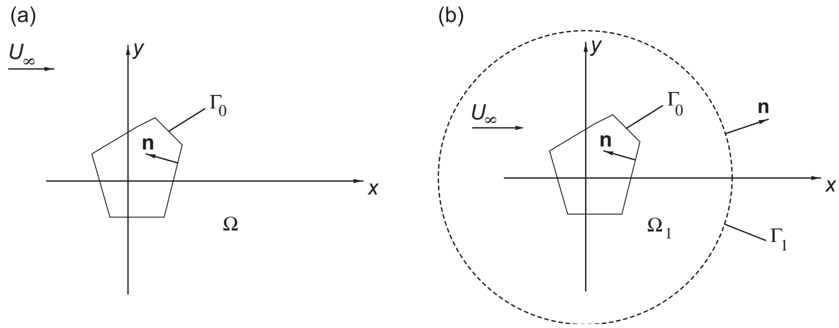

Fig. 1. Schematic of the flow past an obstacle Γ_0 in (a) an unbounded exterior domain Ω and (b) an exterior domain Ω_1 with an outer boundary Γ_1.

where $\mathbf{u} = [u_1, u_2, u_3]$ is the velocity field, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ is the vorticity, p is the pressure, ν represents the coefficient of the kinematic viscosity (the density of the fluid is assumed equal to unity), $\mathbf{u}_0$ is the initial condition, T represents the end of the time interval considered and $\mathbf{x} = [x_1, x_2, x_3]$ is the position vector. Given an object with a boundary Γ_0 characterized by the local unit normal vector $\mathbf{n}$ facing into the object (Fig. 1(a,b)), the hydrodynamic force acting on this object is, by definition, given by the following expression:

$$\mathbf{F} = \mathbf{F}^p + \mathbf{F}^\nu = \oint_{\Gamma_0} p\mathbf{n}\,\mathrm{d}\sigma - \nu\oint_{\Gamma_0}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right]\mathbf{n}\,\mathrm{d}\sigma = \oint_{\Gamma_0} p\mathbf{n}\,\mathrm{d}\sigma + \nu\oint_{\Gamma_0} \mathbf{n} \times \boldsymbol{\omega}\,\mathrm{d}\sigma. \tag{2}$$

The velocity gradient is defined as $[\boldsymbol{\nabla}\mathbf{u}]_{ij} = \partial u_i/\partial x_j$ and the two forms of the viscous term $\mathbf{F}^\nu$ are equivalent due to the identity $\oint_{\Gamma_0}(\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\mathbf{n}\,\mathrm{d}\sigma = 0$ valid for all incompressible fields $\mathbf{u}$. The arguments that we will elaborate in this paper will be valid in both 2D and 3D domains; for the sake of simplicity of exposition, however, the main proof will be restricted to the 2D case with its generalization to 3D being quite straightforward.

It is often convenient to solve equations of fluid motion (1) in one of the so-called "non-primitive" formulations involving only vorticity and velocity, or streamfunction [see, e.g., Gresho (1991), Quartapelle (1993)]. In such cases one does not have direct access to the pressure required to evaluate $\mathbf{F}^p$. Similar situation arises also in experimental investigations where the Particle Image Velocimetry (PIV) measurements are capable of extracting instantaneous velocity and vorticity fields with systematically increasing resolution in space and time [see, e.g., Rockwell (2000)]. Unavailability of pressure in such approaches motivates the need for alternative ways of calculating the hydrodynamic force in which pressure is not needed. In the literature several methods have been proposed, all relying on suitable manipulation of the Navier–Stokes system (1). Below we will briefly review the most important results; derivation of some of these approaches will be analysed in detail in the following section. We also remark that, in view of the assumptions made, these expressions will not include terms corresponding to the motion of the obstacle. This lack of generality, however, does not affect the main point of the paper.

The best-known approach, popularized by Saffman (1992), expresses the force in terms of the vorticity impulse as

$$\mathbf{F} = -\frac{1}{D-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \mathbf{x} \times \boldsymbol{\omega}\,\mathrm{d}\Omega, \tag{3}$$

where $D = 2, 3$, is the spatial dimension. While providing an interesting insight into the relationship between the force and vorticity dynamics, this approach has the disadvantage that integration is extended over the whole infinite domain. Consequently, vorticity at very large distances from the obstacle must be included which can be quite difficult in both numerical simulations and PIV measurements. In addition, the time derivative present in Eq. (3) tends to amplify noise. As an alternative, Noca et al. (1997, 1999) proposed a family of formulas with the generic form

$$\mathbf{F} = -\frac{1}{D-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1} \mathbf{x} \times \boldsymbol{\omega}\,\mathrm{d}\Omega + [\text{integral over } \Gamma_1] + [\text{integral over } \Gamma_0], \tag{4}$$

where integration is restricted to the truncated domain Ω_1 and the far field contribution is contained in the integral over Γ_1. These formulas no longer require integration over an infinite domain, but still suffer from the presence of the time derivative. Furthermore, evaluation of the fluxes involved in the integrals over Γ_1 may be complicated.

A different approach was proposed by Quartapelle and Napolitano (1983) where, before integrating over the domain, the momentum equation (1a) is multiplied by the gradient $\boldsymbol{\nabla}\eta_a$ of a harmonic function η_a which satisfies a Neumann-type boundary condition $\mathbf{n} \cdot \boldsymbol{\nabla}\eta_a = -\mathbf{n} \cdot \mathbf{a}$ on Γ_0 and whose gradient decays to zero at the outer boundary. As a result,