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The dynamic process of the inflation of thin elastomeric shells under the action of an excess pressure☆

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ABSTRACT

A problem of the dynamic process of their deformation is formulated in the momentless approximation for thin shells made of rubber-like elastomers under the action of a time-varying excess hydrostatic pressure. A system of non-linear equations of motion is set up for the case of arbitrary displacements and deformations in which the true deformation of the transverse compression of the shell, corresponding to the use of the modified Kirchhoff–Love model proposed earlier, and the coordinates of the points of the middle surface with respect to a fixed Cartesian system of coordinates, are taken as the required unknown functions. Physical relations connecting the components of the true internal stresses with the elongation factors and the extent of the shear strain are constructed using relations proposed earlier by Chernykh. A finite-difference method is developed for solving the initial-boundary value problem and, on the basis of this, the dynamic process of the inflation of shells of revolution at different rates of pressure increase is investigated and the unstable stages of their deformation are established with a determination of the corresponding limiting (critical) pressure value. After this value has been reached, a further increase in the deformations occurs at decreasing values of the internal pressure.

Products that are thin-walled shells made of highly elastic materials (a synthetic elastomer, latex film or natural rubber) and subjected to considerable deformations (a relative elongation of up to 1000%) during use are widely and diversely applied: catheters used in medicine, car air bags, air balloons, etc. As a rule, calculations of the strength of such constructional components must be based on the use of the relations of the non-linear mechanics of deformable solids and thin shells for finite displacements and deformations. There is an extensive literature [1–12] dealing with the construction of one version or another of these relations. Examples of their application in solving certain problems in the mechanics of elastomers have been presented, in particular, in a monograph.[3] The physical relations constructed in it, relating the components of the true stresses to the components of the true strains in the form of elongation factors have been used [11] to formulate and solve problem of the inflation and static stability of a cylindrical shell with closed ends made of rubber and under the action of an internal pressure. A characteristic feature of this problem is the separation of the process of loading the shell into two stages: in the first stage, an increase in the diameter and length of the shell only occurs when the pressure increases and, in the second stage, a further increase in the above dimensions of the shell and a decrease in its thickness occurs by pumping air into the shell with decreasing pressure. The mechanical explanation of this process involves the onset of the static instability of the rubber shell under conditions of biaxial asymmetric stretching, similar to the formation of a neck in cylindrical samples made of elastoplastic materials when they are stretched [10] in an axial direction under a static load.

The purpose of this paper is to study the loading of thin elastomer shells with an internal pressure described above within the limits of a dynamic formulation of the problem, that extends the results of the investigations carried out earlier.[4,7,10,11] It follows, starting out from an analysis of the results obtained earlier,[11] that taking account of the deformation of the compression of the shell in the transverse direction, the finiteness of the components of the true deformations, the introduction of the true stresses according to Novoshilov [1] and the use of constitutive relations linking the true stresses and true strains with one another is of fundamental importance in its formulation.

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1. The equations of motion of a momentless shell

We assume that, at the instant $t = t_0$, the space V_0 of the undeformed shell is referred to a system of curvilinear coordinates α^1, α^2, z which is normally associated with the middle surface σ_0 that has the principal basis vectors $\mathbf{r}_i^0 = \partial\mathbf{r}^0/\partial\alpha^i$ and components of the principal metric tensor $G_{ij}^0 = \mathbf{r}_j^0\mathbf{r}_j^0$. In the system of coordinates adopted, the radius vector of an arbitrary point $M_0 \in V_0$ is defined by the equality (henceforth Latin indices have the values 1,2 and Greek indices have the values 1, 2, 3)

$$\mathbf{R}^0(\alpha^i, z) = \mathbf{r}^0(\alpha^i) + z\mathbf{e}_3^0, \quad -h/2 \le z \le h/2 \tag{1.1}$$

where $\mathbf{r}^0 = \mathbf{r}^0(\alpha^i)$ is the radius vector of a point on the surface σ_0, h is the initial thickness of the shell, and $\mathbf{e}_3^0$ is the vector of the unit normal to the surface σ_0 that constitute a right handed trihedron with the unit vectors $\mathbf{e}_i^0 = \mathbf{r}_i^0/\sqrt{G_{ii}^0}$.

When the shell is dynamically deformed, we shall define the radius vector of the above-mentioned point $M_0 \in V_0$, that has passed at the instant t to the point $M(\alpha^i, z) \in V$, according to the modified Kirchhoff–Love theory,[11] by the representation

$$\mathbf{R} = \mathbf{R}^0 + \mathbf{U} = \mathbf{r} + z(1+\varphi)\mathbf{e}_3, \quad \mathbf{U} = \mathbf{u} + z[(1+\varphi)\mathbf{e}_3 - \mathbf{e}_3^0] \tag{1.2}$$

where $\mathbf{u} = u^\alpha\mathbf{e}_\alpha^0$ is the vector of the displacements of the points of the middle surface σ_0, $\varphi(\alpha^i)$ is a transverse deformation function that is introduced into the treatment in terms of which the elongation factor λ_3 and the true strain ε_3 in the transverse direction z are defined using the formulae [11]

$$\lambda_3 = 1 + \varphi, \quad \varepsilon_3 = \varphi \tag{1.3}$$

and $\mathbf{e}_\alpha$ are unit vectors in the deformed surface σ with a radius vector $\mathbf{r}$, to determine which we have the formulae

$$\begin{gathered}\mathbf{e}_i = \mathbf{r}_i/A_i, \quad A_i = \sqrt{G_{ii}}, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2\sqrt{G_{11}G_{22}}/\sqrt{G} \\ G = G_{11}G_{22} - G_{12}^2, \quad G_{ij} = \mathbf{r}_i\mathbf{r}_j, \quad \mathbf{r}_i = \partial\mathbf{r}/\partial\alpha^i\end{gathered} \tag{1.4}$$

Note that the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are directed along the tangents to the coordinate lines α^i in the deformed state and $\mathbf{e}_3$ is directed along the normal to the surface σ. The covariant components of the strain tensor

$$\varepsilon_{ij} = (G_{ij} - G_{ij}^0)/2 \tag{1.5}$$

that serve for the calculating of the elongation factors λ_1 and λ_2 in the direction of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and the shear measure $\sin\gamma_{12}$ [9,11] in accordance with the expressions (φ_{12} is the angle between the basis vectors $\mathbf{r}_1^0$ and $\mathbf{r}_2^0$ in the undeformed state)

$$\lambda_i = 1 + \varepsilon_i = \sqrt{1 + 2\varepsilon_{(ii)}}, \quad \sin\gamma_{12} = \frac{2\varepsilon_{(12)}}{\lambda_1\lambda_2\sin\varphi_{12}}, \quad \varepsilon_{(ij)} = \frac{\varepsilon_{ij}}{\sqrt{G_{ii}^0G_{jj}^0}} \tag{1.6}$$

where ε_i is the relative elongation and $\varepsilon_{(ij)}$ is the dimensionless value of the covariant components of the strain tensor, are determined by the difference between the components of the metric tensors G_{ij} and G_{ij}^0.

Assuming that it is a thin momentless shell, in the sections $\alpha^i = \text{const}$ and $z = \text{const}$ of the deformed shell which, at the instant t, has a thickness [11]

$$h_* = h(1 + \varepsilon_3) = h(1 + \varphi) = h\lambda_3$$

we introduce the vectors of the true stress σ^i and σ_3 into the treatment, defining them by the representations

$$\sigma^i = \sigma^{ij}\mathbf{e}_j, \quad \sigma^3 = \sigma^{33}\mathbf{e}_3 \tag{1.7}$$

in which the quantities σ^{ij} and σ^{33} are physical components.

Integrating expression (1.7) over the thickness of the shell h_*, we obtain

$$\mathbf{T}^i = T^{ij}\mathbf{e}_j, \quad \mathbf{T}^3 = T^{33}\mathbf{e}_3 \tag{1.8}$$

where

$$T^{ij} = h\lambda_3\sigma^{ij}, \quad T^{33} = h\lambda_3\sigma^{33} \tag{1.9}$$

We will now assume that surface forces p^- and p^+, applied to points of the faces $z = -h_*/2$ and $z = h_*/2$, as well as a mass force $\mathbf{Q}$ act on an infinitesimal element of thickness h_* separated from the shell and on the surface σ, that is, an infinitesimal area $d\sigma = \sqrt{G}d\alpha^1d\alpha^2$. We will assume that they are defined in the form

$$\mathbf{p} = p\mathbf{e}_3d\sigma = p\mathbf{e}_3\sqrt{G}d\alpha^1d\alpha^2, \quad \mathbf{Q} = \mathbf{g}h_*\rho dF = \mathbf{g}h_*\rho\sqrt{G}d\alpha^1d\alpha^2 \tag{1.10}$$

where $p = p^- + p^+$ is the excess pressure acting on the shell divided by the unit of area $d\sigma$, ρ is the density of the shell material that we shall subsequently consider as invariable when treating shells made of an incompressible elastomer, and $\mathbf{g}$ is the gravitational acceleration.

In the approximation of momentless theory, the transverse internal stress formed in the shell T^{33} and the projection of the principal moment of the external forces in the direction of the normal $\mathbf{e}_3$ that, in the case considered, is equal to

$$M^3 = (p^+ - p^-)h_*$$