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Semi-exact solution of non-uniform thickness and density rotating disks. Part II: Elastic strain hardening solution

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ABSTRACT

Analytical solutions for the elastic–plastic stress distribution in rotating annular disks with uniform and variable thicknesses and densities are obtained under plane stress assumption. The solution employs a technique called the homotopy perturbation method. A numerical solution of the governing differential equation is also presented based on the Runge–Kutta's method for both elastic and plastic regimes. The analysis is based on Tresca's yield criterion, its associated flow rule and linear strain hardening. The results of the two methods are compared and generally show good agreement. It is shown that, depending on the boundary conditions used, the plastic core may contain one, two or three different plastic regions governed by different mathematical forms of the yield criterion. Four different stages of elastic–plastic deformation occur. The expansion of these plastic regions with increasing angular velocity is obtained together with the distributions of stress and displacement.

1. Introduction

Estimation of stress distribution in rotating disks at high speeds is an important subject due to a large number of engineering applications. For this reason, the theoretical investigation of stresses and displacement in such structures has been receiving considerable attention and the topic is discussed in many standard and advanced textbooks [1–4].

Gamer [5] first obtained a consistent analytical solution for the elastic–plastic response of a rotating uniform thickness solid disk using Tresca's yield condition and its associated flow rule. A state of plane stress and linear strain hardening were assumed. It was shown that the plastic core first develops at the axis of the disk and consists of two adjacent plastic regions governed by different mathematical forms of the yield criterion. Güven extended these works to annular and solid disks of variable thickness and variable density [6,7] and to fully plastic variable thickness solid disks with constant thickness in the central portion [8]. The complete analytical solutions for convex exponential and power function thickness profiles were obtained by Eraslan and Orcan [9]. In a later work, Eraslan and Orcan studied the elastic–plastic deformation of variable thickness solid disks having concave profiles [10].The elastic–plastic response of the concave profiles were shown to be quite different from that of the uniform thickness disk. It was shown that the deformation behavior of a concave exponential solid disk is different from that of the constant thickness disk in such a way that three different stages of elastic–plastic deformation take place. The numerical von Mises solution of an exponential solid disk has been compared to the analytical Tresca's solution at the fully plastic state [11]. Various thickness profiles including hyperbolic, exponential and power function forms for annular disks were studied numerically by Eraslan [12]. Elastic–plastic deformations of rotating variable thickness annular disks with free, pressurized and radially constrained boundary conditions were also obtained by Eraslan [13]. Variational iteration solution of elastic non-uniform thickness and density rotating disks by Hojjati and Jafari [14], theoretical and numerical analyses of rotating discs of non-uniform thickness and density of elastic linear hardening material by the variable material property method by Hojjati and Hassani [15] are some of newly published studies in this field. Adomian's decomposition and homotopy perturbation methods have also been used by Hojjati and Jafari [16] for the solution of elastic non-uniform thickness and density rotating disks.

The aim of this work is to obtain semi-analytical elastic–plastic solutions of rotating annular disks with variable thicknesses and densities subjected to different boundary conditions of engineering interest using the homotopy perturbation method (HPM). Perturbation method is one of the well-known methods based on the

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Nomenclature

r	Radius of disk (m)
a	Inner radius of disk (m)
b	Outer radius of disk (m)
c_i	Integration constant
E	Modulus of elasticity (GPa)
E_t	Tangent modulus (GPa)
η	Hardening parameter
ε_{EQ}	Equivalent plastic strain
$h_\circ$	Thickness of the disk at $r = b$ (m)
$h(r)$	Disk thickness
n, m	Parameters in the thickness and density functions
u	Radial displacement
μ	Poisson's ratio
ρ	Mass density (kg/m^3)
σ_r	Radial stress (MPa)
σ_θ	Circumferential stress (MPa)
$\sigma_\circ$	Yield stress of the material (MPa)
ω	Constant angular velocity of rotation (rad/s)
Ω	Dimensionless angular velocity
Γ	The boundary of the domain Θ
Θ	Domain
ε	Small parameter

existence of small/large parameters, the so-called perturbation quantity. He's homotopy perturbation method which does not require a "small parameter" takes full advantage of the traditional perturbation methods and the homotopy techniques and yields a very rapid convergence of the solution. The homotopy perturbation method is discussed in detail by He [17–20].

2. Theoretical background

2.1. Governing equation of rotating disk

Assuming that the stresses do not vary over the thickness of the disk, the analysis used for thin disks of constant thickness can be extended to disks of variable thickness. Let h be the thickness of the disk, varying with radius r, i.e. $h = h(r)$ (Fig. 1). Simplifying the radial equilibrium condition for an infinitesimal element of the disk gives [2,15]:

$$\frac{d}{dr}(hr\sigma_r) - h\sigma_\theta = -h\rho\omega^2 r^2 \tag{1}$$

where $\rho = \rho(r)$ is the distribution function of density, and ω is the angular velocity of the disk.

In the plane stress and small deflection condition assumed for this analysis, the strain–displacement relation is [2]:

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \frac{u}{r} \tag{2}$$

In this work, an elastic linear hardening [2] model is used (Fig. 2) for modeling the stress–strain relationship of the disc material:

$$\begin{cases} \varepsilon = \frac{\sigma}{E}, & \sigma \le \sigma_\circ \\ \varepsilon = \frac{\sigma_\circ}{E} + \frac{1}{E_t}(\sigma - \sigma_\circ), & \sigma > \sigma_\circ \end{cases} \tag{3}$$

where $\sigma_\circ$ and E_t are the yield strength of the material and tangent modulus, respectively. By using this model,

$$\begin{cases} \sigma \le \sigma_\circ & \varepsilon_e = \frac{\sigma}{E}, \quad \varepsilon_p = \varepsilon_{EQ} = 0 \\ \sigma > \sigma_\circ & \varepsilon_e = \frac{\sigma}{E}, \quad \varepsilon_p = \varepsilon_{EQ} = \left(\frac{1}{E_t} - \frac{1}{E}\right)(\sigma - \sigma_0) \end{cases} \tag{4}$$

where ε_{EQ}, ε_p and ε_e are the equivalent plastic, plastic and elastic strains respectively.

By using Eqs. (3) and (4), one can easily get the following relationship for the equivalent stress σ_{EQ} as occurs in the plastic region:

$$\sigma_{EQ} = \sigma_\circ(1 + \eta\varepsilon_{EQ}) \tag{5}$$

where $\eta = EE_t/[\sigma_\circ(E - E_t)]$ represents the hardening parameter.

In the analysis of elastic–plastic response of annular disk, Tresca's yield criterion and its associated flow rule as defined in Section 4.1 are used with the assumption of linear hardening material behavior. The disk is symmetric with respect to the mid plane and its profile and density are assumed to vary as functions of the radius (r) [6,7]:

$$h(r) = h_\circ\left(\frac{r}{b}\right)^{-n} \tag{6}$$

$$\rho(r) = \rho_\circ\left(\frac{r}{b}\right)^{m} \tag{7}$$

where n and m are geometric parameters ($0 \le n \le 1, m \ge 0$), b is the outer radius of the disk and h_0 is the thickness of the disk at $r = b$. It is obvious that a disk of uniform thickness and density is simply obtained by setting $m = n = 0$. Rotating disks with variable density can be considered as functionally graded materials (FGM) and the literature devoted to this field is extensive [6,7,21–24]. Although, other material properties such as the modulus of elasticity can also be assumed to vary with disk radius, in this research it was arbitrarily decided to assume that Young's modulus is constant as reported in previous studies [6,7,16]. The same procedure is also applicable if the variation of the other material properties are to be taken into account.

2.2. Homotopy perturbation method (HPM) [16]

To explain the homotopy method [17–20], let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Theta \tag{8}$$

with the boundary conditions of:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \tag{9}$$

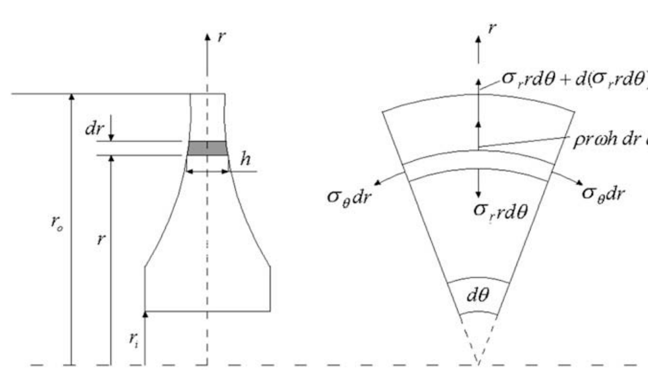

Fig. 1. Disk profile and acting forces on an element.