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Resonance rotations of a pendulum with a vibrating suspension☆

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ABSTRACT

Resonance rotations of the form $s{:}n$ (where s and n are integers), for which the pendulum executes exactly s rotations during n oscillations of the suspension point are investigated for a plane pendulum with a vibrating suspension point, when there is external and internal damping. It is shown analytically that asymptotically stable resonance rotations of the form 1:2, 1:3, 2:1, 2:3 and 3:1 exist. Many other resonance rotations of higher orders are found from the results of a numerical experiment.

A large number of publications are devoted to different aspects of the problem of the motions of a pendulum with a vibrating suspension point. A detailed bibliography can be found from the list of publications given below.[1–11] In the majority of these publications the oscillations of a pendulum in the neighbourhood of a certain fixed direction are investigated. In this paper, using the methods described earlier,[3,4] we investigate the resonance rotations of a pendulum of the form $s{:}n$ (where s and n are integers), for which the pendulum executes exactly s rotations during n oscillations of the suspension point. It was established in Ref. 4, that, for a high frequency and a small amplitude of the oscillations of the suspension point, and when there is a damping moment along the pendulum axis, there are 1:1 and 1:2 stable resonance rotations. Here we investigate other resonance rotations of the pendulum when there is both external and internal dissipation.

1. A plane pendulum with external damping

We will consider a plane pendulum, the suspension point of which vibrates in the form $x = x(t)$, $y = y(t)$, the function $x(t)$ specifies the deflection of the suspension axis with respect to the horizontal, and $y(t)$ specifies the deflection with respect to the vertical. We will denote by l the distance from the suspension point to the mass centre of the pendulum, by A the moment of inertia of the pendulum with respect to the suspension axis, and by φ the angle of deflection of the radius vector of the centre of mass of the pendulum from the downward vertical.

Lagrange's function of the pendulum, apart from an additive function of time, is given by the expression

$$L = T - \Pi = \frac{A\dot{\varphi}^2}{2} + ml\dot{\varphi}(\dot{x}\cos\varphi + \dot{y}\sin\varphi) + mgl\cos\varphi$$

For the case when a damping moment $Q_\psi = -\lambda\dot{\varphi}$ acts along the pendulum axis, denoting by $\omega_0 = \sqrt{mgl/A}$ the frequency of small oscillations of the pendulum when there is no vibration and dissipation, we obtain the equation of motion of the pendulum in the form

$$\varphi'' = -\sin\varphi - \mu\varphi' - \frac{ml}{A}(x''\cos\varphi + y''\sin\varphi) \tag{1.1}$$

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A prime denotes a derivative with respect to dimensionless time $\tau = \omega_0 t$, while $\mu = \lambda/(A\omega_0)$ is the dimensionless damping factor.

We will consider the case when the suspension point performs rectilinear harmonic oscillations at an angle β to the horizontal:

$$x = a\sin\Omega\tau\cos\beta, \quad y = a\sin\Omega\tau\sin\beta \tag{1.2}$$

Here Ω is the ratio of the oscillation frequency of the suspension point to the frequency ω_0. Equation (1.1) then takes the form

$$\varphi'' = -\sin\varphi - \mu\varphi' + \alpha\Omega^2\sin\Omega\tau\cos(\varphi - \beta) \tag{1.3}$$

where $\alpha = mla/A$ is the ratio of the oscillation amplitude of the suspension point to the given length of the pendulum.

To determine the resonance rotation modes of the pendulum of the form $s{:}n$, where s and n are integers, we will change to a new dimensionless time t and to a new variable X, as given by the formulae

$$t = \tau\Omega/n, \quad \varphi - \beta = X + st$$

Equation (1.3) can then be rewritten in the form

$$\ddot{X} = \alpha n^2\sin nt\cos(st + X) - \frac{n}{\Omega}\mu(\dot{X} + s) - \frac{n^2}{\Omega^2}\sin(st + X + \beta) \tag{1.4}$$

Assuming $\Omega \gg 1$ and choosing $\varepsilon = \sqrt{n/\Omega}$ as the small parameter, we obtain the following system in standard form:

$$\dot{X} = \varepsilon Y, \quad \dot{Y} = \varepsilon f(X, Y, t, \varepsilon) \tag{1.5}$$

Here $f(X, Y, t, \varepsilon)$ is a periodic function of the time t with period 2π:

$$f(X, Y, t, \varepsilon) = f_1(X, t) - \mu s - \varepsilon\mu Y - \varepsilon^2 f_2(X, t) \tag{1.6}$$

$$f_1 = p\sin nt\cos(st + X) = \frac{p}{2}\sum_{+,-}\sin[(n \pm s)t \pm X]$$
$$f_2 = \sin(st + X + \beta); \quad p = n\alpha\Omega \tag{1.7}$$

Here we have introduced the notation

$$\sum_{+,-}\sin[(n \pm s)t \pm X] = \sin[(n + s)t + X] + \sin[(n - s)t - X]$$

Similar notation will be used below.

We will investigate the solution of system (1.5) by the averaging method, which necessitates imposing a limit on the function $f(X, Y, t, \varepsilon)$ as $\varepsilon \to 0$. This condition will be satisfied if we put

$$p = n\Omega\alpha \le 1 \tag{1.8}$$

But another way of choosing the small parameter is possible. If we put $\alpha \ll 1$ (note that this condition follows from inequality (1.8) and $\Omega \gg 1$) and choose $\varepsilon_1 = n\sqrt{\alpha} = \varepsilon\sqrt{p}$ as the small parameter, system (1.5) can be rewritten in the form

$$\dot{X} = \varepsilon_1 Y, \quad \dot{Y} = \varepsilon_1 f^*, \quad f^* = \frac{f_1}{p} - \frac{\mu(s + \varepsilon_1 Y)}{p} - \varepsilon_1^2\frac{f_2}{p^2} \tag{1.9}$$

With this choice of the small parameter the function f^* must be limited as $\varepsilon_1 \to 0$, which is ensured by the inequality $p = n\Omega\alpha \ge 1$, which is the opposite of inequality (1.8).

Hence, if we put

$$\Omega \gg 1, \quad \alpha \ll 1 \tag{1.10}$$

it is not necessary to impose any other limitations on the value of Ω and α (in the form of conditions, imposed on the quantity p), since the solutions obtained from systems (1.5) and (1.9), written in terms of Ω and α, will be exactly identical.

Note also that no assumptions have been made regarding the smallness of the damping factor μ at this stage.

Following the previously proposed method,[4] we will construct an averaged system for system (1.5). We obtain the replacement of variables

$$X = x + \sum_{j=1}\varepsilon^j u_j, \quad Y = y + \sum_{j=1}\varepsilon^j \upsilon_j \tag{1.11}$$

as a result of which system (1.5) is reduced to the following autonomous form (apart from terms of order ε^{N+1}):

$$\dot{x} = \varepsilon y, \quad \dot{y} = \sum_{k=1}^{N}\varepsilon^k\Phi_k(x, y) \tag{1.12}$$

It is assumed that the functions $u_j(x, y, t)$ and $v_j(x, y, t)$ are periodic in time t with period 2π and have zero means. The order of averaged system (1.12) will be limited by the number N, for which this system allows of non-degenerate equilibria $(x_0, 0)$ and enables us to draw a conclusion on the nature of their stability, without taking into account higher approximations. Then the required resonance rotations of the pendulum will correspond to these equilibria of system (1.12) and, using Bogolyubov's second theorem,[4] one can draw conclusions on their stability.