Journal of Applied Mathematics and Mechanics 79 (2015) 539–545

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The dynamics of a disk on a rough inclined plane when there is an uneven normal stress distribution☆

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ARTICLE INFO

Article history:
Received 18 September 2014
Available online 17 May 2016

ABSTRACT

The motion of a uniform circular disk of non-zero height on a fixed rough inclined plane is considered on the assumption that the disk moves without losing contact, resting on the plane with its own base. The friction forces and moment are calculated using a model of the contact stress distribution, including three independent parameters. During the translational motion of the disk, the normal pressure distribution corresponds to the normal stress distribution on the base of a punch with a flat base and, for zero height of the disk, agrees with Galin's law. A qualitative analysis of the dynamics of the disk in the case when the slope of the plane is less than the Coulomb friction coefficient is given.

As previously,[1] in the problem considered here the normal pressure distribution on the base of the disk depends on three independent parameters, and the results obtained for a punch with a flat base[2,3] are taken into account. The problem of the motion of a flat disk on a horizontal plane with a uniform normal pressure distribution was considered in Ref. 4. The motion of a disk of non-zero height along an inclined plane was investigated in Ref. 5 in the dynamically consistent model of friction. The contract stress distribution assumed in this paper for a zero height of the disk is not, as previously in Ref. 5, uniform, but agrees with the Galin distribution.[3]

1. Statement of the problem

We will consider the continuous plane-parallel motion of a uniform disk of mass m, radius a and height $2h$ on a fixed rough inclined plane, on which the base of the disk rests. The velocity of the mass centre $\mathbf{u}$ is parallel to the inclined plane, while the angular velocity $\boldsymbol{\omega}$ is perpendicular to this plane.

Using the fundamental theorems of mechanics, we can write the equations of motion of the disk on the inclined plane in the form

$$m\dot{\mathbf{u}} = m\mathbf{g} + \mathbf{N} + \mathbf{F}, \quad ma^2\dot{\boldsymbol{\omega}}/2 = \mathbf{M} + \mathbf{M}_N \tag{1.1}$$

Here $\mathbf{g}$ is the gravitational acceleration, $\mathbf{N}$ is the normal reaction of the inclined plane, $\mathbf{F}$ is the resultant friction force, acting on the disk, and $\mathbf{M}$ and $\mathbf{M}_N$ are the moments of the friction and normal reaction forces, calculated with respect to the centre of mass of the disk.

We will introduce a right orthonormalized frame of reference $C\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, such that C is the centre of the base of the disk, the unit vector $\mathbf{e}_1$ is directed along the velocity $\mathbf{u} = u\mathbf{e}_1 (u \geq 0)$ of the point C, the unit vector $\mathbf{e}_2$ is orthogonal to the velocity $\mathbf{u}$ and, like the unit vector $\mathbf{e}_1$, lies in the inclined plane, while the unit vector $\mathbf{e}_3$ is directed along the normal to the plane of motion. Then $\omega = \omega\mathbf{e}_3$.

The position of an arbitrary point A of the base of the disk is defined by the angle $\beta \in [0, 2\pi]$ between the vectors $\mathbf{e}_1$ and $\mathbf{CA}$ and the distance $r = CA$.

The forces $\mathbf{N}$ and $\mathbf{F}$, and also the moments $\mathbf{M}$ and $\mathbf{M}_N$ are calculated from the formulae

$$\mathbf{N} = \mathbf{e}_3\iint n(A)dS, \quad \mathbf{F} = -\iint kn(A)\frac{\mathbf{v}(A)}{|\mathbf{v}(A)|}dS$$

☆ *Prikl. Mat. Mekh.*, Vol. 79, No. 6, pp. 768–777, 2015.
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http://dx.doi.org/10.1016/j.jappmathmech.2016.04.012
0021-8928/

$$\mathbf{M} = -\iint\left[\mathbf{r}(A) \times kn(A)\frac{\mathbf{v}(A)}{|\mathbf{v}(A)|}\right]dS, \quad \mathbf{M}_N = \iint n(A)[\mathbf{r}(A) \times \mathbf{e}_3]dS$$

The integration is carried out over the base of the disk, $n(A)$ is the normal pressure density at the point A, $k>0$ is the coefficient of sliding friction, and $v(A)$ is the velocity of the point A, which can be calculated from Euler's formula

$$\mathbf{v}(A) = \mathbf{u} + [\boldsymbol{\omega} \times \mathbf{CA}] = (u - \omega r \sin\beta)\mathbf{e}_1 + \omega r \cos\beta\mathbf{e}_2$$

2. Model of the normal pressure distribution

To calculate the forces and moments of the friction, it is necessary to determine the normal pressure density $n(A)$. We will assume that

$$n(A) = \frac{\lambda_0 + \lambda_1 r\cos\beta + \lambda_2 r\sin\beta}{\sqrt{a^2 - r^2}}$$

For translational motion of the disk, this distribution has the same form as the normal pressure distribution over the base of a plane punch, indented into an elastic half-plane.[2] As previously,[1] the coefficients λ_0, λ_1 and λ_2 are determined at each instant from the relations which ensure continuous motion of the disk:

$$(\mathbf{N} + \mathbf{F}, \mathbf{e}_3) = 0, \quad (\mathbf{M} + \mathbf{M}_N, \mathbf{e}_i) = 0, \quad i = 1, 2 \tag{2.1}$$

It follows from the first relation of (2.1) that $\lambda_0 = (2\pi a)^{-1} mg\cos\alpha$, while the coefficients λ_1 and λ_2 are found as the solution of the system of two linear equations

$$a_{11}\lambda_1 + a_{12}\lambda_2 = a_{10}\lambda_0, \quad a_{21}\lambda_1 + a_{22}\lambda_2 = a_{20}\lambda_0 \tag{2.2}$$

where

$$a_{11} = -2\pi a^2 kh\int_0^1 \frac{s^3}{1-s^2}\left\langle \frac{\cos^3\beta}{D(s, \beta; z)}\right\rangle ds$$

$$a_{22} = 2\pi a^2 kh\int_0^1 \frac{s^2}{1-s^2}\left\langle \frac{(z - s\sin\beta)\sin\beta}{D(s, \beta; z)}\right\rangle ds$$

$$a_{12} = -a_{21} = \frac{2\pi a^3}{3}, \quad a_{10} = 0, \quad a_{20} = -2\pi akh\int_0^1 \frac{s}{1-s^2}\left\langle \frac{z - s\sin\beta}{D(s, \beta; z)}\right\rangle ds$$

$$s = \frac{r}{a}, \quad z = \frac{u}{\Omega}, \quad \Omega = a\omega, \quad D(s, \beta; z) = \sqrt{z^2 - 2zs\sin\beta + s^2}$$

$$\langle F(s, \beta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} F(s, \beta)d\beta$$

Note that $a_{11} < 0$, $a_{22} < 0$. Consequently, $\Delta = a_{11}a_{22} - a_{12}a_{21} > 0$, and so system (2.2) has the unique solution

$$\lambda_1 = -a_{12}a_{20}\lambda_0/\Delta, \quad \lambda_2 = a_{11}a_{20}\lambda_0/\Delta$$

Henceforth it will be more convenient to use the following notation:

$$\tilde{n}(s, \beta) = 1 + \tilde{\lambda}_1 s\cos\beta + \tilde{\lambda}_2 s\sin\beta, \quad n(s, \beta) = \frac{\lambda_0 \tilde{n}(s, \beta)}{a\sqrt{1 - s^2}}$$

$$\tilde{\lambda}_i = a\lambda_i/\lambda_0, \quad i = 1, 2$$

It should be noted that

$$\tilde{\lambda}_i = \tilde{\lambda}_i(z, \delta), \quad \delta = 3kh/a, \quad i = 1, 2$$

Since $n(A)$ is the normal pressure density at the point A of the contact point, for all points of the base of the disk the inequality $n(A) \geq 0$ must be satisfied.