Journal of Applied Mathematics and Mechanics 79 (2015) 440–445

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Bifurcations of the relative equilibria of a heavy bead on a hoop uniformly rotating about an inclined axis with dry friction☆

D.V. Balandin, E.S. Shalimova

N. I. Lobachevskii State University of Nizhny Novgorod, Nizhny Novgorod
M.V. Lomonosov Moscow State University, Moscow, Russia

ARTICLE INFO

Article history:
Received 23 December 2014
Available online 6 May 2016

ABSTRACT

The sliding of a heavy bead threaded onto a thin circular hoop rotating at a constant angular velocity about an inclined axis located in its plane and passing through its centre is considered. A dry friction force acts between the bead and the hoop. The sets of the non-isolated positions of relative equilibrium of the bead on the hoop are found and their dependence on the parameters of the problem is investigated. The results are presented in the form of bifurcation diagrams.

Problems similar to that considered here arise in the dynamics of mechanical systems with rotating parts performing different operations such as the mixing, grinding, drying, etc. of diverse substances (see Refs 1, 2, for example) as well as self-compensating systems.[3]

In the one-dimensional case, the study of the dynamics of such systems probably dates back to Mandelshtam's papers,[4] where he refers to different versions of a Froud pendulum. As a rule, it is assumed in studying a Froud pendulum that the resisting force is independent of the normal reaction. However, generally speaking, this is not so in the case of the motion of a system with dry friction. General methods have been developed (Refs 5–7 etc.) for studying limiting cycles of a system of the Froud pendulum type under quite general assumptions.

The existence of non-isolated sets of equilibria of systems with friction has been known for a long time (for example, see Refs 8, 9). The investigation of the stability of non-isolated equilibria in such systems in rigid-body mechanics probably originates in Krementulo's papers.[10,11] A general theory of the stability of the equilibria in systems with dry friction has been developed.[12] Methods, based on the general theory of systems with discontinuous right-hand sides, have been proposed for studying the stability of such equilibria.[13–15] Bifurcations of the equilibria in systems with friction have been studied as well as bifurcations of the phase portraits of such systems.[16–18]

The bifurcation sets in the problem of the motion of a heavy bead on a circular hoop rotating about its own vertical diameter have been considered[19] and, in a similar problem, for the case when the axis of rotation is vertical but does not coincide with the axis of symmetry of the hoop.[20] Unlike in these problems, the case of an inclined axis of rotation of the hoop passing through its centre is considered below.

1. Statement of the problem and equations of motion in redundant coordinates

The motion of a heavy particle, that is, a bead P of mass m threaded onto a hoop in the form of a circle of radius ℓ with its centre at the point O, is considered. The hoop rotates with constant angular velocity ω about an inclined axis lying in its plane and passing through its centre. The angle of inclination of the axis from the vertical is assumed to be constant and equal to α. A dry friction force with a coefficient of friction μ acts between the bead and the hoop.

The motion of the bead can be described using Lagrange's equations of the first kind in a moving coordinate system (MCS) associated with the hoop. Suppose $Oxyz$ is a right-handed triplet with origin at the centre of the hoop, the z axis of which is directed along its axis of rotation, the y axis is located in the plane of the hoop and the x axis is perpendicular to this plane (see Fig. 1).

In the MCS, the bead position P is given by the coordinates (x, y, z) and the constraints restricting its motion are defined by the relations

$$f_1 = \frac{1}{2}(y^2 + z^2 - \ell^2) = 0, \quad f_2 = x = 0 \tag{1.1}$$

☆ *Prikl. Mat. Mekh.*, Vol. 79, No. 5, pp. 627–634, 2015.
E-mail address: ekateryna-shalimova@yandex.ru (E.S. Shalimova).

http://dx.doi.org/10.1016/j.jappmathmech.2016.03.004
0021-8928/

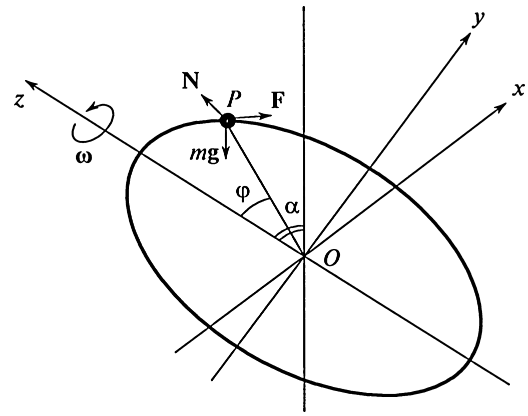

Fig. 1.

Suppose $\boldsymbol{v}_r = (\dot{x}, \dot{y}, \dot{z})$ is the bead velocity in the MCS, $\upsilon_r = (\upsilon_r, \upsilon_r)^{1/2}$, and the transfer velocity $\upsilon_e = (-\omega y, \omega x, 0)$. The kinetic energy of the system, free from constraints, and the potential energy in the MSC are given by the relations

$$T = \frac{1}{2}m((\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2)$$

$$U = mg(x\sin\omega t\sin\alpha + y\cos\omega t\sin\alpha + z\cos\alpha)$$

where g is the gravitational acceleration. Lagrange's equations

$$\frac{d}{dt}\frac{\partial L_\lambda}{\partial \dot{x}} = \frac{\partial L_\lambda}{\partial x}, \quad \frac{d}{dt}\frac{\partial L_\lambda}{\partial \dot{y}} = \frac{\partial L_\lambda}{\partial y} + F_y, \quad \frac{d}{dt}\frac{\partial L_\lambda}{\partial \dot{z}} = \frac{\partial L_\lambda}{\partial z} + F_z \tag{1.2}$$

where

$$L_\lambda = L + \lambda_1 f_1 + \lambda_2 f_2, \quad L = T - U \tag{1.3}$$

and $\mathbf{F} = (0, F_y, F_z)$ is the friction force, can be represented in the form

$$m\mathbf{a} = \mathbf{F}_C + \mathbf{F}_c + \mathbf{F}_N + \mathbf{N} + \mathbf{F}$$

Here $\mathbf{a}$ is the acceleration of the bead in the MSC, $\mathbf{F}_C$ and $\mathbf{F}_c$ are the Coriolis force and the centrifugal force, $\mathbf{F}_N$ is the gravitational force and $\mathbf{N}$ is the normal reaction of the hoop. The unit vectors

$$\boldsymbol{\tau} = \begin{pmatrix} 0 \\ -z/\ell \\ y/\ell \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 0 \\ -y/\ell \\ -z/\ell \end{pmatrix}, \quad \mathbf{b} = \boldsymbol{\tau} \times \mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

respectively define the tangent, internal normal and binormal to the circle at the point P. The expressions for the forces have the form

$$\mathbf{F}_C = \begin{pmatrix} 0 \\ -2m\omega\dot{y} \\ 2m\omega\dot{x} \end{pmatrix}, \quad \mathbf{F}_c = \begin{pmatrix} 0 \\ m\omega^2 y \\ m\omega^2 z \end{pmatrix}, \quad \mathbf{F}_N = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

$$\mathbf{N} = -\lambda_1 \ell \mathbf{n} + \lambda_2 \mathbf{b} = \begin{pmatrix} \lambda_2 \\ \lambda_1 y \\ \lambda_1 z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 \\ F_y \\ F_z \end{pmatrix}, \quad \mathbf{F} \parallel \boldsymbol{\tau}$$

and, in the case of slipping $\boldsymbol{v}_r \neq 0$

$$\mathbf{F} = -\mu\frac{\boldsymbol{v}_r}{\upsilon_r}N, \quad N = (\mathbf{N}, \mathbf{N})^{1/2} \tag{1.4}$$

We introduce dimensionless parameters using the relations

$$x \mapsto x\ell, \quad y \mapsto y\ell, \quad z \mapsto z\ell, \quad t \mapsto t\sqrt{\frac{\ell}{g}}, \quad \omega \mapsto \omega\sqrt{\frac{g}{\ell}}$$

$$\lambda_1 \mapsto \lambda_1 m\frac{g}{\ell}, \quad \lambda_2 \mapsto \lambda_2 mg, \quad L \mapsto Lmg\ell, \quad F_y \mapsto mgF_y, \quad F_z \mapsto mgF_z \tag{1.5}$$