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New kinematic parameters of the finite rotation of a rigid body[☆]

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ABSTRACT

A new family of kinematic parameters for the orientation of a rigid body (global and local) is presented and described. All the kinematic parameters are obtained by mapping the variables onto a corresponding orientated subspace (hyperplane). In particular, a method of stereographically projecting a point belonging to a five-dimensional sphere $S^5 \subset R^6$ onto an orientated hyperplane R^5 is demonstrated in the case of the classical direction cosines of the angles specifying the orientation of two systems of coordinates. A family of global kinematic parameters is described, obtained by mapping the Hopf five-dimensional kinematic parameters defined in the space R^5 onto a four-dimensional orientated subspace R^4. A correspondence between the five-dimensional and four-dimensional kinematic parameters defined in the corresponding spaces is established on the basis of a theorem on the homeomorphism of two topological spaces (a four-dimensional sphere $S^4 \subset R^5$ with one deleted point and an orientated hyperplane in R^4). It is also shown to which global four-dimensional orientation parameters–quaternions defined in the space R^4 the classical local parameters, that is, the three-dimensional Rodrigues and Gibbs finite rotation vectors, correspond. The kinematic differential rotational equations corresponding to the five-dimensional and four-dimensional orientation parameters are obtained by the projection method. All the rigid body kinematic orientation parameters enable one, using the kinematic equations corresponding to them, to solve the classical Darboux problem, that is, to determine the actual angular position of a body in a three-dimensional space using the known (measured) angular velocity of rotation of the object and its specified initial position.

1. Formulation of the problem

Suppose a rigid body with a fixed point, that coincides with the origin of the inertial system of coordinates $oX_1X_2X_3$ at the first instant, is rigidly associated with a right-handed moving system of coordinates $oY_1Y_2Y_3$. We shall consider the spherical motion of a body around a fixed point and set up the problem of determining the spatial position of such a body in the space R^3 using the new global orientation parameters.

From a mathematical point of view the problem must be regarded as methods of embedding the $SO(3)$ manifold (the set of all rotations of a rigid body) in the n-dimensional space R^n. This problem of determining the orientation of a body in R^3 as a problem of the parametrization of the $SO(3)$ group has been studied previously and described in detail (see Refs 1 and 2, for example). The basic methods for the global one-to-one parametrization of the $SO(3)$ group of rotations have also bee analyzed and described. [3,4] Note that, besides the classical parametrization methods that are nine-dimensional (using the orthogonal 3×3 matrix $A \equiv X$ of the direction cosines) and six-dimensional (using two columns of the same matrix), a modified six-dimensional parametrization is known [3] that is introducible for describing the little-known Hopf five-dimensional parametrization.

The $SO(3)$ configuration space formed by all possible positions of a rigid body with one fixed point in the case of a standard embedding in the space R^6 defines an open set of points located on the sphere $S^5 \subset R^6$. Denoting this space by $M^5 (M^5 \subset S^5)$, we have a corresponding topological space M^*. Actually, the $SO(3)$ group is closely associated with the five-dimensional sphere $S^5 \subset R^6$. In particular, M^* is the set of points of the surface of the sphere S^5 and simultaneously an orientated subspace, that is, the manifold $SO(3) \subset S^5$. We shall also assume that the sphere S^5 is imbedded in the space R^6 in the standard way, that is, as a set of points defined by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 1 \tag{1.1}$$

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Here, the set M^* associates a certain five-dimensional orientation vector with each point of the sphere S^5 that can be drawn through any two points of its open surface.

Below we consider new kinematic parameters for the orientation of a rigid body, obtained by the method of stereographic projection of the points of the spheres $S^5 \subset R^6$ and $S^4 \subset R^5$ (and the vectors identical to them) onto orientated hyperplanes of the spaces R^5 and R^4 respectively.

2. Five-dimensional sphere, six- and five-dimensional orientation parameters

In order to define the new kinematic parameters, we will briefly recall the modified six-dimensional parametrization and the procedure for imbedding the $SO(3)$ group into the five-dimensional space R^5 (see Refs 3 and 4).

We introduce a six-dimensional vector as the column vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6)^T$, the elements of which consist of the first two columns of the renormalized orientation matrix $A \equiv X$, that is, the column vector $\tilde{x} = X/\sqrt{2}$. Here, the matrix $X \in SO(3) \subset R^6$.

Since the rotation matrix $A \equiv X$ considered is orthogonal $X^T \equiv X^{-1}$, for its elements it is possible to construct six known [1,3] independent constraint conditions that take account of the orthogonality of the basis axes of the body in a trihedron. Using these conditions, introducing specially renormalized direction cosines of the angles specifying the orientation of the two basis vectors (the two trihedra corresponding to the systems of coordinates introduced above) and taking account of the normalization $X/\sqrt{2}$, we obtain the equalities

$$\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 = 1/2, \quad \tilde{x}_4^2 + \tilde{x}_5^2 + \tilde{x}_6^2 = 1/2, \quad \tilde{x}_7^2 + \tilde{x}_8^2 + \tilde{x}_9^2 = 1/2 \tag{2.1}$$

It is well known that only six of the direction cosines of the angles introduced above are required for the global one-to-one parametrization. Discarding the last equation in constraint equations (2.1) and adding the first two equations, we can represent them in the form

$$\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 + \tilde{x}_4^2 + \tilde{x}_5^2 + \tilde{x}_6^2 = 1 \tag{2.2}$$

Analysis of the constraint conditions presented above shows that the specially renormalized parameter vector $\tilde{x} \in R^6$ corresponding to equality (2.2) essentially remains the same orientation vector $x \in R^6$, the elements of which are the classical direction cosines of the angles although it corresponds to different normalization conditions, that is, (2.1) or (2.2). The vector $\tilde{x}$ therefore consists of six renormalized elements, that is the orientation parameters (the standard direction cosines $x_{ik}(i, k = 1, 2, 3)$ of the angles of the two system of coordinates) and the following matrix identities hold for it [4]

$$\tilde{x}^T\tilde{x} = 1, \quad \tilde{x}^T\mathbf{J}_i\tilde{x} = 0, \quad i = 1, 2; \quad J_1 = \begin{Vmatrix} E_3 & O_3 \\ O_3 & -E_3 \end{Vmatrix}, \quad J_2 = \begin{Vmatrix} O_3 & E_3 \\ E_3 & O_3 \end{Vmatrix} \tag{2.3}$$

where E_3 is the unit matrix and O_3 is a null 3×3 matrix. In equalities (2.3), the scalar product $\tilde{x}^T\tilde{x} = 1$ and it is the set of points of a five-dimensional sphere S^5 of unit radius embedded in the space R^6 in the standard way.

In the space R^6 of the (specially) renormalized parameters $\tilde{x} \in (S^5 \backslash \{\alpha\})$ considered here, a straight line can be drawn through two specified points, the "north pole" N and the current point $\tilde{x}\,(\tilde{x} = (\tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3,\ \tilde{x}_4,\ \tilde{x}_5,\ \tilde{x}_6))$ on the sphere S^5 up to its intersection with the orientated hyperplane $P\{(\tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3,\ \tilde{x}_4,\ \tilde{x}_5) \in R^5; (\tilde{x}_6 = 0)\}$. This means that a stereographic projection of the point $\tilde{x}$ exists together with a modified vector $\tilde{x} \in M^* \subset S^5$ corresponding to this point in the orientated hyperplane $P \subset R^5$. The point of intersection of the line with this hyperplane is the required five-dimensional orientation vector $y \in P$, that satisfies the following projection equation [3,4]

$$y = \frac{V\tilde{x}}{1 - \alpha^T\tilde{x}} \tag{2.4}$$

The projection operator V is a rectangular 5×6 matrix that can be represented in the form of two submatrices: $V = ||E_5 0_{5\times1}||$ (E_5 is a unit 5×5 matrix).

Hence, Eq. (2.4) gives the point of intersection of the line joining a specific point $N(0, 0, 0, 0, 0, 1)$ of the stereographic projection and an arbitrary point $\tilde{x} \in M^*$ on the sphere (of unit radius) $S^5 \subset R^6$ with the hyperplane $P \subset R^5$ that is orientated in the subspace R^5.

The inverse mapping is the inversion or projection of the five-dimensional vector y, as a point belonging to the orientated hyperplane $P \subset R^5$, onto the sphere $S^5 \subset R^6$. It is precisely inversion that enables one directly to find the six-dimensional vector $\tilde{x} \in R^6$ introduced above for which the inverse projective transformation equation

$$\tilde{x} = \frac{\alpha(y^Ty - 1) + 2V^Ty}{y^Ty + 1} \tag{2.5}$$

holds [4]

It is necessary here to mention briefly a specific geometric interpretation of Eq. (2.5). The matrix vector product V^Ty is a six-dimensional vector that is orthogonal to the radius passing through the "centre" of the stereographic projection, that is, the "north pole" N. The line joining the points V^Ty and N (the centre of the projection) intersects the sphere S^6 at a certain point $\tilde{x} \in M^*$ ($M^* \subset R^6$ is an open set). In this case, if $\tilde{x}$ is a specially renormalized six-dimensional vector $\tilde{x} \in M^*$, then the required five-dimensional orientation vector $y \in R^5$ must satisfy the two constraint equations [4]

$$\alpha^TJ_i\alpha(y^Ty - 1)^2 + 4y^TVJ_i\alpha(y^Ty - 1) + 4y^TVJ_iV^Ty = 0, \quad i = 1, 2 \tag{2.6}$$

where J_i are the matrices defined above in relations (2.3).