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Reverse motions of mechanical systems☆

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ABSTRACT

The possibility of the occurrence of sections of reverse motions in natural mechanical systems, when, in the second half of a time interval, the motion in the first half of the interval is repeated in the reverse order and the opposite velocity with a specified accuracy, is investigated. It is shown that such motions are characteristic of natural mechanical systems in the neighbourhood of a non-degenerate equilibrium position if the natural frequencies are independent. Systems with gyroscopic and dissipative forces are also considered. It is shown that, in these systems, sections of reverse motion can be observed in a special system of coordinates. Examples are presented.

1. Reverse motions

Consider a natural mechanical system with a smooth Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}) = T - V$, where $\mathbf{q} \in R^n$ are generalized coordinates, $T(\mathbf{q}, \dot{\mathbf{q}})$ is the kinetic energy which is a quadratic form in the generalized velocities, and $V(\mathbf{q})$ is the potential energy.

We shall say that the motion $\mathbf{q}(t)$ is a reverse motion in a section $t_1 \le t \le t_2$ if, in this section, the system passes through the same positions as before the instant t_1 but in the reverse order and with the opposite velocity: $\mathbf{q}(t) = \mathbf{q}(2t_1 - t)$ when $t_1 \le t \le t_2$. Using the notation $\boldsymbol{\upsilon}(t) = \dot{\mathbf{q}}(t)$, we see that $\boldsymbol{\upsilon}(t) = \boldsymbol{\upsilon}(2t_1 - t)$.

It has been pointed out[1] that, if the generalized velocity vanishes at the instant t_1, the motion of the system is a reverse motion in the section $t \ge t_1$. Actually, if $\mathbf{q}(t)$ is a motion of the system, then $\tilde{\mathbf{q}}(t) = \mathbf{q}(2t_1 - t)$ is also a motion. At the instant t_1, the initial conditions of these motions are identical: $\mathbf{q}(t) = \tilde{\mathbf{q}}(t_1)$, $\dot{\mathbf{q}}(t_1) = \dot{\tilde{\mathbf{q}}}(t_1) = 0$. The reversibility condition $\mathbf{q}(t) = \tilde{\mathbf{q}}(t)$ is satisfied when $t \ge t_1$ by virtue of the uniqueness of the solutions of the equations of motion.

We shall say that a motion $\mathbf{q}(t)$ of a system is a reverse motion to within $\varepsilon > 0$ in a sectioin $t_1 \le t \le t_2$ if the reversibility condition is satisfied in this section to within ε:

$$\left|\mathbf{q}(t) - \mathbf{q}(2t_1 - t)\right| + \left|\mathbf{v}(t) + \mathbf{v}(2t_1 - t)\right| < \varepsilon$$

The constant of the energy integral is denoted by h: $T + V = h$. We will fix h and only consider motions with an energy h. We shall assume that the boundary of the domain of possible motion (DPM) $\Gamma(h) = \{\mathbf{q} : V(\mathbf{q}) = h\}$ is compact and we will denote the distance from it to a point $\mathbf{q}$ by $\rho(\mathbf{q}, \Gamma)$.

If the system falls on the boundary of the DPM at the instant t_1: $V(\mathbf{q}(t_1)) = h$, then $\dot{\mathbf{q}}(t_1 = 0)$ and the motion is also a reverse motion in the section $t \ge t_1$.

Assertion 1. For a specified value of the energy h and for any $\varepsilon, T > 0$, a $\delta(\varepsilon, \mathrm{T}, \mathrm{h}) > 0$ exists such that, if $q(t)$ is a motion with energy h and $\rho(\mathbf{q}(t_1), \Gamma(h) < \delta)$ or $|\dot{q}(t_1| < \delta)$, then the motion is a reverse motion to within ε in the segment $t_1 \le t \le t_1 + T$.

Actually, in both cases if δ is a sufficiently small quantity, then the velocity $\dot{\mathbf{q}}(t_1)$ is quite low and the point $\mathbf{q}(t_1)$ is quite close to the boundary of the DPM $\Gamma(h)$. A point $\mathbf{q}^* \in \Gamma(h)$ which is fairly close to $\mathbf{q}(t_1)$ is therefore found. The motion $\hat{\mathbf{q}}(t)$ with the initial conditions $\hat{\mathbf{q}}(t_1) = \mathbf{q}^* \dot{\hat{\mathbf{q}}}(t_1) = 0$ is a reverse motion when $t \ge t_1$. Its initial conditions differ to a fairly small extent from the initial conditions of the initial motion $\mathbf{q}(t_1)$, $\dot{\mathbf{q}}(t_1)$. The value of δ can be chosen to be so small that, in the section $t_1 - T \le t \le t_1 + T$, the two motions diverge with respect

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to the phase coordinates by no more than $\varepsilon/2$:

$$|\mathbf{q}(t)-\hat{\mathbf{q}}(t)|+|\dot{\mathbf{q}}(t)-\dot{\hat{\mathbf{q}}}(t)|<\varepsilon/2$$

The reversibility condition to within ε for $q(t)$ in the section $t_1 \le t \le t_1 + T$ follows at once from this.

2. Linear systems

Suppose the Lagrangian is quadratic and $\mathbf{q} = 0$ is a stable non-degenerate equilibrium position:

$$L = (\langle A\dot{\mathbf{q}},\dot{\mathbf{q}}\rangle - \langle B\mathbf{q},\mathbf{q}\rangle)/2$$

where A and B are symmetric positive-definite $n \times n$ matrices. The constant of the energy integral h is equal to zero at equilibrium. Motion is only possible for $h > 0$. In the normal coordinates $\xi = \{\xi_1 \dots, \xi_n\}$, the equations of motion have the form

$$\ddot{\xi}_i = -\omega_i^2\xi_i, \quad i = 1,\dots,n \tag{2.1}$$

where ω_i are the natural frequencies of the system, the squares of which are the roots of the characteristic equation $\det(-\omega_2 A + B) = 0$. The general solution of system (2.1) has the form

$$\xi_i(t) = c_i\cos(\omega_i t + d_i), \tag{2.2}$$

where c_i and d_i are certain constants, $i = 1,\dots, n$.

The natural frequencies of a system are said to be independent if, in the case of the integers $m_1, \dots, m_n$, the condition $\sum m_i\omega_i = 0$ is only satisfied when all the $m_i = 0$.

Assertion 2. Suppose the natural frequencies of the system are independent when, for any $\varepsilon > 0$ and $T > 0$, a value $\Delta(\varepsilon, T) > 0$ (which does not depend on the energy integral constant h) exists such that, for any motion of the system with energy h, there will be an infinite number of sections of reverse motion of length T to within $\varepsilon\sqrt{h}$. At the same time, the origin of at least one section of reverse motion will be located in any time interval of length Δ.

In other words, for any motion $\mathbf{q}(t)$ with an energy h and for any instant t_0, an instant t_1 exists such that $t_0 \le t_1 \le t_0 + \Delta$ and the motion is a reverse motion to within $\varepsilon\sqrt{h}$ in the section $t_1 \le t \le t_1 + T$.

Proof. For any constant λ, if $\mathbf{q}(t)$ is a motion with an energy h, then $\lambda\mathbf{q}(t)$ is a motion with an energy λ^2h. It is therefore sufficient to carry out the proof for the case when $h = 1$. We now transfer to normal coordinates and prove that, for any $\varepsilon > 0$, a $\Delta > 0$ can be found such that, for any motion $\xi(t)$ with an energy $h = 1$ in any time interval $t_0 \le t \le t_0 + \Delta$, an instant t_1 exists when the velocity of the motion is small, that is, $|\dot{\xi}(t_1)| < \varepsilon$. After this, the correctness of Assertion 2 will follow from Assertion 1.

For brevity, we shall write $|a| < \varepsilon \bmod 2\pi$ in the case when a point on a circle with an angular coordinate a is located in the ε-neighbourhood of the zero point.

Since the motion has the form (2.2), it is sufficient to show that, for any $\varepsilon > 0$, a $\Delta > 0$ exists found such that, for any values of d_i in the interval $0 \le t \le \Delta$, an instant t_1 is found such that $|\omega_i t_1 + d_i| < \varepsilon \bmod 2\pi$ for all $i = 1,\dots, n$. For the proof, we consider a conditioned-periodic motion in an n-dimensional torus with frequencies $(\omega_1, \dots, \omega_n)$. We take a trajectory passing through the point 0 at the instant $t = 0$. Since the frequencies are independent, it covers the torus densely everywhere.[2] This means that a $\Delta > 0$ exists such that a part of this trajectory in the time interval $0 \le t \le \Delta$ will be spaced from any point of the torus by not more than ε. This value of Δ will be the required value.

Note that the quantity Δ can be estimated much better (generally speaking, it can be 2^n times smaller) if it is taken into account that, in order for the velocity of the motion to be small, it can be required that not only do the quantities $\omega_i t_1 + \varphi_i$ fall within the neighbourhood of zero but also within the neighbourhood of π.

We also point out that the accuracy $\varepsilon\sqrt{h}$ in Assertion 2 can be replaced by εd, where d is the diameter of the DPM at an energy level h. This follows from the fact that $d = c\sqrt{h}$, where c is a constant which depends solely on the matrix B.

3. General case

Suppose $\mathbf{q} = 0$ is a stable non-degenerate equilibrium position of a natural mechanical system with Lagrangian

$$L = (\langle A\dot{\mathbf{q}},\dot{\mathbf{q}}\rangle - \langle B\mathbf{q},\mathbf{q}\rangle)/2 + O((|\mathbf{q}|+|\dot{\mathbf{q}}|)^3)$$

where A and B are symmetric positive-definite $n \times n$ matrices. At equilibrium, the energy integral constant is equal to zero. Motion is only possible for $h > 0$. We shall examine the motion of a system for fairly small values of this constant: $h_1 \le h_0$, such that the component of the connectivity of the DPM containing the point 0 is compact.

Assertion 3. A positive constant $h_1 \le h_0$ is found such that Assertion 2 is satisfied for any motion with an energy $h < h_1$.

Proof. Suppose $\mathbf{q}(t)$ is the motion of a system and $\mathbf{q}^*(t)$ is the motion of the linearized system with the same initial conditions:

$$\mathbf{q}^*(t_0) = \mathbf{q}(t_0), \quad \dot{\mathbf{q}}^*(t_0) = \dot{\mathbf{q}}(t_0)$$

The system is linearized at the equilibrium position, and the right-hand sides of the equations of motion in the linear and the general case therefore differ by the quantity $\delta = O(|\mathbf{q}|^2)$ when $h \to 0$. Since the diameter of the DPM is of the order of $\sqrt{h}$, then $\delta = O(h)$. Hence, during a time $T_0 = \Delta + T$, the deviation of $\mathbf{q}(t)$ from $\mathbf{q}^*(t)$ will be of the order of $\exp(cT_0)\delta = O(h)$ ($c > 0$ is the constant from the Gronwall–Bellman