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Parameter dependence and stability of guided TE-waves in a lossless nonlinear dielectric slab structure

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ABSTRACT

The nonlinear Schrödinger equation is the basis of the traditional stability analysis of nonstationary guided waves in a nonlinear three-layer slab structure. The stationary (independent of the propagation distance) solutions of the nonlinear Schrödinger equation are used as ”initial data” in this analysis. In the present paper, we propose a method to investigate the dependence of these solutions on the experimental parameters and discuss their stability with respect to the parameters. The method is based on the phase diagram condition (PDC) and compact representation (in terms of Weierstrass' elliptic function and its derivative) of the dispersion relation (DR). The problem's parameters are constrained to certain regions in parameter space by the PDC. Dispersion curves inside (or at boundaries) of these regions correspond to possible physical solutions of Maxwell's equations as ”start” solutions for a traditional stability analysis. Numerical evaluations of the PDC, DR, and power flow including their parameter dependence are presented.

1. Introduction

Investigating stability of electromagnetic fields in waveguides (linear or nonlinear represents an important issue in physics and applied mathematics. In nonlinear waveguide theory, stability of the transverse electric field $\mathcal{E}_y(x,z,t)$ (in a planar three-layer structure (see Fig. 1)) usually is studied by assuming an ansatz (to solve Maxwell's equations)

$$\mathcal{E}_y(x,z,t) = \tilde{E}_y(x,z,\gamma)e^{i(\gamma z-\omega t)}, \tag{1}$$

leading to Helmholtz equations for $\tilde{E}_y(x,z,\gamma)$ valid in the three layers. If the nonlinear part of the permittivity is small compared with the linear one the well-known [1] “Slowly Varying Envelope Approximation” ($|\partial_z \tilde{E}_y| \ll |\gamma \tilde{E}_y|$) can be applied. It approximates the Helmholtz equations by nonlinear Schrödinger equations (NLSEs)

$$2i\gamma\frac{\partial \tilde{E}_y}{\partial z} + \frac{\partial^2 \tilde{E}_y}{\partial x^2} - (\gamma^2 - \epsilon(\tilde{E}_y^2))\tilde{E}_y = 0, \tag{2}$$

which are used as the basis for a stability analysis with respect to the propagation distance z. From the physical point of view the stability problem is how an initial field profile (at $z = 0$, say) $\tilde{E}_y(x,0,\gamma)$ evolves with increasing z. The propagation constant γ, as solution of the dispersion relation (DR), depends on the thickness h of the film and on the (material) parameters of the problem. A stability analysis that yields (e.g.) a stable (z-independent) profile $E_y(x,\gamma_0)$ with a certain propagation constant γ_0, may lead, for slightly different parameters, to a propagation constant γ not in the vicinity of γ_0. In general, this implies that the profile $E_y(x,\gamma_0)$ and $E_y(x,\gamma)$ differ considerably, indicating instability. Thus, obviously, it is important, for physical application, to know the parameter dependence of the propagation constant γ. From the mathematical point of view the stability problem is studied by introducing a perturbation function $f(x,z,\gamma)$ and setting $\tilde{E}_y(x,z,\gamma) = E_y(x,\gamma) + f(x,z,\gamma)$. The NLSE (2) can be linearized leading to a system of two coupled differential equations [2] that are rewritten as an eigenvalue problem for $f(x,z,\gamma)$ (with two operators). The growth rate of $f(x,z,\gamma)$ is studied by analysing the spectrum of the operators [2,3]. For discrete positive imaginary eigenvalues, $f(x,z,\gamma)$ decays to zero as $z \to \infty$ [2], so that, in this sense, $E_y(x,\gamma)$ is stable. It is important to note that in this approach, applied in numerous articles [2–17], the propagation constant γ is obtained by assuming $\tilde{E}_y(x,z,\gamma)$ constant with respect to z, i.e., $\tilde{E}_y(x,z,\gamma) = E_y(x,\gamma)$ [2]. Hence, the eigenvalues and thus the growth rate of $f(x,z,\gamma)$ depend on $E_y(x,\gamma)$. As is well known (see, e.g., [18,19]), $E_y(x,\gamma)$ can be singular or a (real) propagation constant γ, may not exist for certain parameters $p_i \in \{h, J_0, \bar{\epsilon}_v\}$. Furthermore, needless to say, that the decay properties of the perturbation eigenfunction f depends on the parameters p_j.

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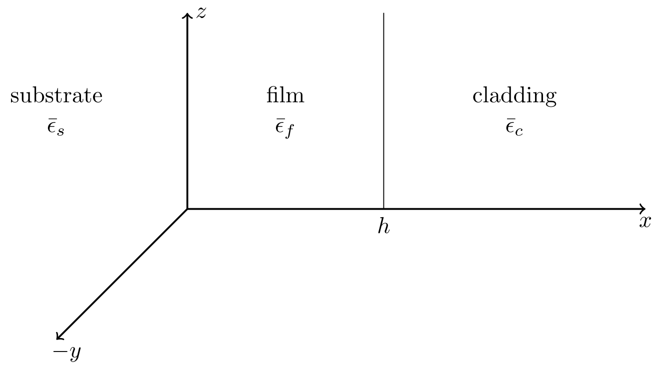

Fig. 1. Geometry of the problem.

Based on the phase diagram analysis [20] conditions for solvability and existence of solutions $\gamma(p_i;p_j)$ [21] of the DR have been derived [22(b)]. With respect to the traditional stability analysis, sketched before, they are appropriate to describe nonnegative and bounded ("physical") intensities $J(x,\gamma)=E_y^2(x,\gamma)$. In connection with stability, there are three possibilities for $\gamma=\gamma(p_i;p_j)$ to depend on a variation δp_j. First, the resulting mode $\tilde{\gamma}(p_i;p_j+\delta p_j)$ can be inconsistent with the PDC. Second, $\tilde{\gamma}(p_i;p_j+\delta p_j)$ may satisfy the PDC with $\tilde{\gamma}\nrightarrow\gamma$ as $\delta p_j\to 0$. Third, again with PDC satisfied, $\tilde{\gamma}\to\gamma$ as $\delta p_j\to 0$.

The paper is organized as follows. Section 2 presents the problem. Section 3 is devoted to the solutions of the basic nonlinear differential equations, and to the dispersion relations with its solvability conditions. The results of Section 3 are applied for certain parameters ϵ_ν, a_ν, h in Section 4. The paper concludes with a summary and some remarks in Section 5.

2. Statement of the problem

We consider a planar waveguide structure with lossless, isotropic, nonmagnetic homogeneous material and a permittivity ϵ according to

$$\epsilon=\begin{cases}\bar{\epsilon}_s=\epsilon_s+a_s|\mathcal{E}_y|^2, & x<0,\\ \bar{\epsilon}_f=\epsilon_f+a_f|\mathcal{E}_y|^2, & 0\le x\le h,\\ \bar{\epsilon}_c=\epsilon_c+a_c|\mathcal{E}_y|^2, & x>h,\end{cases} \tag{3}$$

with $\epsilon_\nu, a_\nu, \nu=s,f,c$, (see Fig. 1) real and constant.

Assuming $\tilde{E}_y(x,z,\gamma)$ and propagation constant γ to be real and inserting ansatz (1) with $\tilde{E}_y(x,z,\gamma)=E_y(x,\gamma)$ into Maxwell's equations we obtain

$$\frac{d^2E_y(x,\gamma)}{dx^2}=\begin{cases}(\gamma^2-\omega^2\bar{\epsilon}_s\mu_0)E_y(x,\gamma), & x<0,\\ (\gamma^2-\omega^2\bar{\epsilon}_f\mu_0)E_y(x,\gamma), & 0\le x\le h,\\ (\gamma^2-\omega^2\bar{\epsilon}_c\mu_0)E_y(x,\gamma), & x>h.\end{cases} \tag{4}$$

With $k_0^2=\omega^2\mu_0\epsilon_0$ and scaling γ and x by k_0, ϵ by ϵ_0, Helmholtz Eq. (4) can be written as

$$\frac{d^2E(x,\gamma)}{dx^2}=\begin{cases}(\gamma^2-\bar{\epsilon}_s)E(x,\gamma), & x<0,\\ (\gamma^2-\bar{\epsilon}_f)E(x,\gamma), & 0\le x\le h,\\ (\gamma^2-\bar{\epsilon}_c)E(x,\gamma), & x>h,\end{cases} \tag{5}$$

where $E(x,\gamma)$ denotes $E_y(x,\gamma)$ with scaled arguments. Inserting $\bar{\epsilon}_\nu$ according to Eqs. (3) into Eqs. (5), multiplying by E', and integrating (with respect to E) we get

$$\left(\frac{dJ(x,\gamma)}{dx}\right)^2=\begin{cases}-2a_sJ^3(x,\gamma)+4(\gamma^2-\epsilon_s)J^2(x,\gamma)+4C_sJ(x,\gamma)\\ \qquad :=R_s(J), \quad x<0,\\ -2a_fJ^3(x,\gamma)+4(\gamma^2-\epsilon_f)J^2(x,\gamma)+4C_fJ(x,\gamma)\\ \qquad :=R_f(J), \quad 0\le x\le h,\\ -2a_cJ^3(x,\gamma)+4(\gamma^2-\epsilon_c)J^2(x,\gamma)+4C_cJ(x,\gamma)\\ \qquad :=R_c(J), \quad x>h,\end{cases} \tag{6}$$

where $J(x,\gamma)=E^2(x,\gamma)$, and C_ν are the integration constants. The problem is, first, to find propagation constant γ associated to physical solutions $J_\nu(x,\gamma)$ of Eqs. (6) that satisfy radiation conditions at infinity

$$E(x,\gamma)\to 0,\quad \frac{dE(x,\gamma)}{dx}\to 0,\quad |x|\to\infty, \tag{7}$$

second, to analyse the parameter dependence of γ, and thus the stability of solutions $J_\nu(x,\gamma)$.

3. Solution

Apart from $J(x,\gamma)=constant$ the solutions of Eqs. (6) are [23,22]:

$$J_{s\pm}(x,\gamma)=\frac{J_0}{\left(\cosh(x\sqrt{\gamma^2-\epsilon_s})\mp\sqrt{1-\frac{a_sJ_0}{2(\gamma^2-\epsilon_s)}}\sinh(x\sqrt{\gamma^2-\epsilon_s})\right)^2},\quad x<0, \tag{8}$$

$$J_{f\pm}(x,\gamma)=J_0+\frac{\frac{1}{2}R_f'(J_0)\left(\wp-\frac{1}{24}R_f''(J_0)\right)\pm\wp'\sqrt{R_f(J_0)}+\frac{1}{24}R_f(J_0)R_f'''(J_0)}{2\left(\wp-\frac{1}{24}R_f''(J_0)\right)^2-\frac{1}{48}R_f(J_0)R_f''''(J_0)},\quad 0\le x\le h,$$

or, equivalently,

$$\begin{aligned}J_{f\pm}(x,\gamma)=J_0&-\frac{9a_fJ_0^2-12(\gamma^2-\epsilon_f)J_0-6C_f}{6\wp(x;g_2,g_3)+3a_fJ_0-2(\gamma^2-\epsilon_f)}\\ &-\frac{18(a_fJ_0^3-2(\gamma^2-\epsilon_f)J_0^2-2C_fJ_0)}{(6\wp(x;g_2,g_3)+3a_fJ_0-2(\gamma^2-\epsilon_f))^2}\\ &\pm\frac{18\wp'(x;g_2,g_3)\sqrt{-2a_fJ_0^3+4(\gamma^2-\epsilon_f)J_0^2+4C_fJ_0}}{(6\wp(x;g_2,g_3)+3a_fJ_0-2(\gamma^2-\epsilon_f))^2},\end{aligned} \tag{9}$$

with g_2 and g_3 according to Eqs. (12), and with derivative

$$\begin{aligned}\partial_xJ_{f\pm}(x,\gamma)=&\frac{\frac{1}{2}R_f'(J_0)\wp'\pm(6\wp^2-\frac{g_2}{2})\sqrt{R_f(J_0)}}{2\left(\wp-\frac{1}{24}R_f''(J_0)\right)^2-\frac{1}{48}R_f(J_0)R_f''''(J_0)}\\ &-\frac{2R_f'(J_0)\wp'\left(\wp-\frac{1}{24}R_f''(J_0)\right)^2\pm4\left(\wp-\frac{1}{24}R_f''(J_0)\right)(\wp')^2\sqrt{R_f(J_0)}}{\left(2\left(\wp-\frac{1}{24}R_f''(J_0)\right)^2-\frac{1}{48}R_f(J_0)R_f''''(J_0)\right)^2}\\ &-\frac{\frac{1}{6}\wp'\left(\wp-\frac{1}{24}R_f''(J_0)\right)R_f(J_0)R_f'''(J_0)}{\left(2\left(\wp-\frac{1}{24}R_f''(J_0)\right)^2-\frac{1}{48}R_f(J_0)R_f''''(J_0)\right)^2},\quad 0\le x\le h,\end{aligned} \tag{10}$$

$$J_{c\pm}(x,\gamma)=\frac{J^{(h)}}{\left(\cosh((x-h)\sqrt{\gamma^2-\epsilon_c})\mp\sqrt{1-\frac{a_cJ^{(h)}}{2(\gamma^2-\epsilon_c)}}\sinh((x-h)\sqrt{\gamma^2-\epsilon_c})\right)^2},\quad x>h, \tag{11}$$

where $\wp(x;g_2,g_3)$ denotes Weierstrass' elliptic function, and J_0, $J^{(h)}$ have been chosen so that intensities are continuous at $x=0$ and $x=h$, respectively. The invariants g_2, g_3 of $\wp(x;g_2,g_3)$ are given by

$$\begin{cases}g_2=2a_fC_f+\frac{4}{3}(\gamma^2-\epsilon_f)^2,\\ g_3=\frac{2}{3}a_fC_f(\epsilon_f-\gamma^2)-\frac{8}{27}(\gamma^2-\epsilon_f)^3.\end{cases} \tag{12}$$

In Eqs. (9) and (10) the prime denotes differentiation with respect to J for $R_f(J)$ and differentiation with respect to x for $\wp(x;g_2,g_3)$. In deriving Eqs. (8) and (11), $C_s=C_c=0$ has been used (according to condition (7) applied to $R_\nu(J), \nu=s,c$, written in terms of E) and, as established