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Propagation of waves from an arbitrary shaped surface—A generalization of the Fresnel diffraction integral

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ABSTRACT

Using the method of Laplace transform the field amplitude in the paraxial approximation is found in the two-dimensional free space using initial values of the amplitude specified on an arbitrary shaped monotonic curve. The obtained amplitude depends on one *a priori* unknown function, which can be found from a Volterra first kind integral equation. In a special case of field amplitude specified on a concave parabolic curve the exact solution is derived. Both solutions can be used to study the light propagation from arbitrary surfaces including grazing incidence X-ray mirrors. They can find applications in the analysis of coherent imaging problems of X-ray optics, in phase retrieval algorithms as well as in inverse problems in the cases when the initial field amplitude is sought on a curved surface.

1. Introduction

Since the pioneering works of Leontovich and Fock [1] the parabolic wave equation (PWE) is widely used in many fields of physical and engineering sciences to describe the propagation of paraxial or quasi-paraxial beams in free space as well as in inhomogeneous media. It was successfully applied for solution of complex problems in laser physics [2], electromagnetic radiation propagation [3], underwater acoustics [4,5], X-ray optics [6], microscopy and lensless imaging [7,8]. This versatile nature of the PWE encourages searching for new applications and new methods of its solution.

One of the areas that can benefit greatly from such new methods is the coherent X-ray imaging [9,10], which has been made possible by the development of powerful, versatile and coherent or quasi-coherent X-ray sources such as laboratory X-ray lasers [11–13], free electron lasers [14] and high order harmonics sources [15]. The coherent X-ray imaging offers several advantages over traditional imaging techniques: a possibility of the lensless imaging and phase retrieval [16], diffraction imaging [17], a sub-picosecond temporal resolution, etc. However, the coherent X-ray imaging, particularly in the reflective mode, poses a number of rather complicated mathematical problems [18,16]. One of them is appropriate description of the radiation field propagation starting from an arbitrary surface, which in general case can be non-flat, off-axis and tilted (see Fig. 1).

It should be noted that one of the underutilized mathematical properties of the PWE is a possibility to express the field amplitude in a part of free space through the initial values of amplitude specified on an arbitrary shaped line or surface. The case of a tilted straight line or plane was studied in [19–22] for applications in grazing incidence reflective microscopy and lithography. The present paper extends the PWE solution to a more general case of the initial surface being an arbitrary shaped one-dimensional monotonic curve. The parabolic initial curve and corresponding exact PWE solution will be also discussed.

2. Direct problem on an arbitrary curve

2.1. General case

Let us consider the 2D PWE for the field amplitude u in coordinates (x, z) [3], where z is the longitudinal coordinate along the beam propagation direction

$$i\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

where it is assumed for simplicity that the wave number $k = 1/2$. Let us assume that u is known at some 2D initial curve (see Fig. 1), which is defined by the following equation

$$x - g(z) = 0, \quad g(0) = 0, \quad u_0(z) = u(g(z), z), \tag{2}$$

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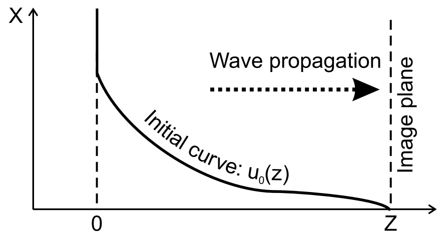

Fig. 1. A scheme showing the position of initial curve with field amplitude values $u_0(z)$ together with the image plane. The coordinate definitions are also shown.

where $g(z)$ is a monotonic positive function. Let us introduce new coordinates z' and x' as

$$\begin{aligned} z' &= z, \\ x' &= x - g(z). \end{aligned} \tag{3}$$

In the new coordinates Eq. (1) can be rewritten as:

$$i\frac{\partial u}{\partial z} = ig'(z)\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} \tag{4}$$

with the initial condition $u(0,z) = u_0(z)$ specified at the line S' ($x' = 0$) parallel to the axis z. In Eq. (4) prime marks of coordinates, for the sake of brevity, were omitted. The Eq. (4) can be solved using the Laplace transform by coordinate x

$$F(w,z) = \int_0^\infty u(x,z)\exp(-wx)\,dx. \tag{5}$$

Applying it to Eq. (4) one can obtain the following differential equation for the function $F(w,z)$

$$iF'_z = ig'(z)wF - ig'(z)u_0 - w^2F + wu_0 + u_1, \tag{6}$$

where the transversal derivative

$$u_1 = u'_x(0,z). \tag{7}$$

A solution of Eq. (6) can be written as

$$\begin{aligned} F(w,z) &= -\int_{-\infty}^{z}\left(g'(z') + iw\right)u_0(z')\exp\left[(iw+G)w(z-z')\right]\,dz' \\ &\quad - i\int_{-\infty}^{z} u_1(z')\exp\left[(iw+G)w(z-z')\right]\,dz', \end{aligned} \tag{8}$$

where

$$G(z,z') = \frac{\int_{z'}^{z} g'(\xi)\,d\xi}{z-z'} = \frac{g(z)-g(z')}{z-z'}. \tag{9}$$

To obtain amplitude u one should apply the reverse Laplace transform

$$u(x,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[wx]F(w,z)\,dw, \quad c \geq 0. \tag{10}$$

Since the expression under integral in (10) must not have any non-regularities in the right semi-plane of w (including the imaginary axis), it is assumed that $c = 0$. The absence of non-regularities is a direct consequence of a, so called, transparent boundary condition [23]. Furthermore, taking into account that

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\exp[wx + (iw+G)w(z-z')]\,dw \\ &= \frac{1}{2\sqrt{\pi i}\sqrt{z-z'}}\exp(i\Phi'), \end{aligned} \tag{11}$$

and that

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} w\exp[wx + (iw+G)w(z-z')]\,dw \\ &= \frac{1}{2\pi i}\frac{\partial}{\partial x}\int_{-i\infty}^{+i\infty}\exp[wx + (iw+G)w(z-z')]\,dw \\ &= \frac{1}{2\sqrt{\pi i}\sqrt{z-z'}}\frac{\partial}{\partial x}\exp(i\Phi') \\ &= \frac{i^{1/2}}{4\sqrt{\pi}\sqrt{z-z'}}\left(G + \frac{x}{z-z'}\right)\exp(i\Phi'), \end{aligned} \tag{12}$$

where

$$\Phi' = \frac{(z-z')}{4}\left(G + \frac{x}{z-z'}\right)^2,$$

and then substituting (11) and (12) into (10) and (8), one obtains the following final expression for the field amplitude u

$$\begin{aligned} u(x,z) &= -\frac{1}{2\sqrt{-\pi i}}\int_{-\infty}^{z}\frac{u_1(z')}{\sqrt{z-z'}}\exp(i\Phi')\,dz' \\ &\quad + \frac{1}{4\sqrt{\pi i}}\int_{-\infty}^{z}(G - 2g'(z'))\frac{u_0(z')}{\sqrt{z-z'}}\exp(i\Phi')\,dz' \\ &\quad + \frac{x}{4\sqrt{\pi i}}\int_{-\infty}^{z}\frac{u_0(z')}{(z-z')^{3/2}}\exp(i\Phi')\,dz'. \end{aligned} \tag{13}$$

Expression (13) depends on two functions – u_0 and u_1 although the initial conditions (2) specify only one of them – u_0. To find u_1 let us assume $x = 0$ in formula (13). In this case, because

$$\lim_{x\to 0} x\int_{-\infty}^{z}\frac{u_0(z')}{(z-z')^{3/2}}\exp(i\Phi')\,dz' = \frac{2\sqrt{\pi}}{\sqrt{-i}}u_0(z), \tag{14}$$

one can obtain the following integral equation of the Volterra first kind for u_1

$$\begin{aligned} u_0(z) &= -\frac{1}{\sqrt{-\pi i}}\int_{-\infty}^{z}\frac{u_1(z')}{\sqrt{z-z'}}\exp(i\Phi)\,dz' \\ &\quad + \frac{1}{2\sqrt{\pi i}}\int_{-\infty}^{z}(G - 2g'(z'))\frac{u_0(z')}{\sqrt{z-z'}}\exp(i\Phi)\,dz', \end{aligned} \tag{15}$$

where

$$\Phi = \frac{(z-z')}{4}G^2, \tag{16}$$

which must be solved before the field amplitude u can be calculated using formula (13).

A problem similar to one discussed here was considered in [24, see p. 521 — boundary problems for regions with moving boundaries]. The method used for PWE solution in the present work allowed one to obtain simpler expressions which have not been explicitly written before.

2.2. Tilted line

Let us consider the case when the initial curve (2) is a tilted (inclined) straight line. One can try to obtain the already known formula to check correctness of the derived result. So, one has

$$const = g'(z) = G(z,z') = -\tan\theta, \tag{17}$$

where θ is the angle between this tilted line and the axis z. Now it follows from (15) that u_1 can be expressed through u_0 as

$$u_1(z) = -\frac{i}{2}\tan\theta - \frac{i^{3/2}}{\sqrt{\pi}}\frac{\partial}{\partial z}\int_{-\infty}^{z}\frac{u_0(\xi)}{\sqrt{z-\xi}}\exp\left[\frac{i}{4}\tan^2\theta(z-\xi)\right]\,d\xi. \tag{18}$$

Expression (18) is (as was said above) the transparent boundary condition for Eq. (4) [23]. Now substituting (18) into (13) and applying the necessary transformations it is possible to show that

$$\begin{aligned} u(x,z) &= \frac{x}{2\sqrt{\pi i}}\int_{-\infty}^{z}\frac{u_0(\xi)}{(z-\xi)^{3/2}} \\ &\quad \times \exp\left[\frac{i(z-\xi)}{4}\left(\tan\theta - \frac{x}{z-\xi}\right)^2\right]\,d\xi. \end{aligned} \tag{19}$$

Taking into account that x in (19) is in reality x' with omitted prime mark, and substituting its expression $x' = x + z\tan\theta$ into (19) it can be obtained that

$$u(x,z) = \frac{x + z\tan\theta}{2\sqrt{\pi i}}\int_{-\infty}^{z}\frac{u_0(\xi)}{(z-\xi)^{3/2}}\exp\left[\frac{i(x+\xi\tan\theta)^2}{4(z-\xi)}\right]\,d\xi. \tag{20}$$